Extensions of the Oxtoby-Ulam Theorem on the prevalence of ergodicity for measure preserving homeomorphisms

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Abstract

While they were Junior Fellows at Harvard in the 1930's, J. C. Oxtoby and S. M. Ulam worked on the conjecture of G. D. Birkhoff and E. Hopf that ergodicity is the 'general case' for homeomorphisms of a manifold preserving a fixed measure. Under the guidance of M. Stone, they proved this conjecture in an important paper published in 1941. Much later A. Katok and A. Stepin extended this result to the property of weak mixing, and afterwards the first author gave combinatorial proofs of both results using a technique of P. Lax.

This article surveys the subsequent work of the authors in extending the result of Oxtoby and Ulam in two different directions. First we describe the work of the first author leading to the conclusion that any property generic for measure preserving bijections of a Lebesgue probability space is also generic for homeomorphisms of a compact manifold preserving a fixed measure. Then we describe the work of both authors in extending the work of Oxtoby and Ulam to noncompact manifolds, with modifications based on the ends of the manifold.

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1 Introduction

1.1 Historical background

Many important physical systems evolve over time according to differential equations which describe certain physical laws. Such systems can often be described by the motion of a point (coding the state of the system) in a finite dimension phase space or, given certain conservation laws, in a lower dimensional manifold of the phase space. Regularities in the differential equations governing the system may be reflected in the continuity of the transformation that describes the change in the system in discrete time intervals. Furthermore, an important result known as Liouville's Theorem ensures that for certain systems there is a measure on the manifold which is invariant under the discrete time transformation. Thus in a natural way one is led to the study of measure preserving homeomorphisms of a manifold.

An important measure theoretic assumption that is often made in the study of such systems is that of ergodicity. A measurable transformation g of a measure space (X, Σ, μ) is said to be *ergodic* (originally the term was *metrically transitive*) if every g-invariant measurable set satisfies either $\mu(A) = 0$ or $\mu(X - A) = 0$. The historical significance of this assumption was explained by J. Oxtoby and S. Ulam in their pioneering paper [55, p. 874] which initiated all the investigations which we will be describing in this article.

> "In the classical theory the assumption was made that the average time spent in any region of phase space is proportional to the volume of the region in terms of the invariant measure, more generally, that time–averages may be replaced by space–averages. To justify this interchange, a number of hypotheses were proposed, variously known as ... quasi–ergodic hypotheses. ... A rigorous discussion of the precise conditions under which the interchange was admissable was only made possible in 1931 by the ergodic theorem of Birkhoff. This established the *existence* of the time–averages in question ... and showed that ... the interchange is permissible if and only if the flow in phase space is *metrically transitive.* A transformation or a flow is metrically transitive if there do not exist two disjoint invariant sets both having positive measure. Thus the effect of the ergodic theorem was to replace the [quasi-]ergodic hypothesis by the hypothesis of metrical transitivity.

Of course invertible ergodic transformations are easy to construct – in some cases even continuous ones. For example, any irrational rotation of a circle is ergodic. However, even on such simple manifolds as the unit cube $[0,1]^n$, $n \geq 2$, it is not easy to explicitly construct an ergodic volume preserving homeomorphism. Consequently mathematicians were worried that the assumption of ergodicity for some of the physical systems mentioned above might involve a topological contradiction - perhaps certain manifolds could not support *any* ergodic measure preserving homeomorphism. On the other hand, both G. D.

Birkhoff and E. Hopf had conjectured that in this context ergodicity was in fact the general case, which if true would eliminate the fear of nonexistence. This was the state of affairs when Oxtoby and Ulam arrived as Junior Fellows at Harvard in the late 1930's. Working together, under the supervision of M. Stone and G. D. Birkhoff, they sought to determine which measured manifolds (X, μ) could support an ergodic μ-preserving homeomorphism. Their main finding (see Theorem 3 for a precise statement) was that for *any* compact manifold X of dimensional at least two, and any nonatomic locally positive Borel measure μ, the ergodic transformations constitute a dense G_δ subset in the space $\mathcal{M}[X, \mu]$ of all μ-preserving homeomorphisms of X, with respect to the uniform topology. Their work on this problem is described in the section '*Harvard Years, 1936-1939'* of Ulam's autobiography [64].

> "In order to complete the foundation of the ideas of statistical mechanics connected with the ergodic theorem, it was necessary to prove the existence, and what is more, the prevalence of ergodic transformations. G. D. Birkhoff himself had worked on special cases in dynamical problems, but there were no general results. We wanted to show that on every manifold (a space representing the possible states of a dynamical system) – the kind used in statistical mechanics – such ergodic behavior is the rule. ... We discussed various approaches to a possible construction of these transformations. ... We kept G.D. Birkhoff informed of the status of our attacks on the problem. ... He would check what I told him with Oxtoby, a more cautious person. It took us more than two years to break through and to finish a long paper [55] which appeared in The Annals of Mathematics in 1941, and which I consider one of the more important results that I had a part in."

A few years after the work of Oxtoby and Ulam, P. Halmos published two papers ([43],[44]) which confirmed the Birkhoff-Hopf conjecture (and more) in a purely measure theoretic context. He proved that ergodicity (and subsequently, that the stronger property of weak mixing) represented the general case in the larger space $\mathcal{G}[X, \mu]$ consisting of all μ-preserving bijections of a Lebesgue probability space (X, μ). That is, he showed that the set of such transformations formed a dense G_δ subset in $\mathcal{G}[X, \mu]$, with respect to the weak topology. (See the next section for all relevant definitions.) Halmos claimed that there were no logical connections between his results and those of Oxtoby and Ulam – see section 1.3 for his claim and our rebuttal.

It was not until 1970 that it was finally established, by A. Katok and A. Stepin [47], that weak mixing homeomorphisms were generic in $\mathcal{M}[X, \mu]$. A few years later the first author ([3], [4], [5]) adapted an argument of P. Lax [49] to give short combinatorial proofs of the Oxtoby-Ulam and Katok-Stepin genericity results for ergodicity and weak mixing in $\mathcal{M}[X, \mu]$. However at this time (1976) no logical connections had yet been established between genericity results in $\mathcal{G}[X, \mu]$ (with the weak topology) and $\mathcal{M}[X, \mu]$ (with the uniform topology), although various authors had remarked on the coincidence that the same properties (including some not mentioned above) seemed to be generic in both contexts.

The purpose of this article is to survey the authors' work over the past quarter century on extensions of the Oxtoby-Ulam theory, that is, on *generic* properties (those containing a dense G_δ set) of homeomorphisms of a manifold X which preserve a fixed measure μ. In particular, our results sometimes imply the *existence* of various properties (fixed points, chaos, etc.) in $\mathcal{M}[X,\mu]$, but we will not cover results where existence is established via means other than genericity. We shall also exclude results which require differentiability.

After completing this introductory section with a subsection on definitions, and one describing the genericity results proved up to 1976, the main portion of this article which surveys our work is divided into two parts. The first part, Section 2, covers the first author's work in extending the Oxtoby-Ulam theory on compact manifolds to further measure theoretic properties. In particular, the coincidence of genericity results in $\mathcal{G}$ and $\mathcal{M}$ is explained by the result (Theorem 23) that any property which is generic for automorphisms of a finite Lebesgue space (with the weak topology) is also generic for homeomorphisms of a compact manifold preserving a fixed measure (with the uniform topology). This synthesis is achieved by taking the 'finite Lebebesgue space' to be the one determined by the given manifold and measure, and then to consider $\mathcal{M}[X,\mu]$ as a subset of $\mathcal{G}[X,\mu]$. An important result used in obtaining this synthesis is the 'Lusin Theorem for Measure Preserving Homeomorphisms' (Theorem 11), which establishes that the embedding $\mathcal{M}[X,\mu] \subset \mathcal{G}[X,\mu]$ is a dense one that has other important properties. A second result that is needed is an extension of the well known Rokhlin Tower Theorem, called the Multiple Tower Rokhlin Theorem (Theorem 19), which in turn shows that (Theorem 21) the conjugates in $\mathcal{G}[X,\mu]$ of any antiperiodic transformation in $\mathcal{G}[X,\mu]$ form a set whose uniform topology closure contains $\mathcal{M}[X,\mu]$. Taken together, Theorems 19 and 21 combine to give a proof of our main genericity theorem (Theorem 23) for properties in $\mathcal{M}[X,\mu]$.

The second part of the article, Section 3, covers the work of both authors in extending the Oxtoby-Ulam theory to noncompact manifolds. This area of investigation was initiated by the second author [59], who proved in 1979 that ergodicity is generic in the space $\mathcal{M}[R^n,\lambda]$ of volume preserving homeomorphisms of Euclidean space. Subsequent work of the authors showed that the measure theoretic behaviour near a homeomorphism $h \in \mathcal{M}[X,\mu]$, for a sigma-compact manifold X, depended crucially on the end structure of X and on the homeomorphism h^* induced by h on the ends of X. (Roughly speaking, and end of X is a way of going to infinity on X; for example Euclidean space R^n has a single end 'at infinity' for $n \geq 2$, but two ends when $n = 1$.) We show in particular (Theorem 25) that h can be approximated by ergodic homeomorphisms (in the compact-open topology) if and only if the end homeomorphism h^* is incompressible and the charge (finitely additive signed measure) that it induces on the ends is identically zero. A consequence of this result is that the Oxtoby-Ulam Theorem (generic ergodicity in $\mathcal{M}[X,\mu]$) holds for a sigma-compact manifold X if and only if X has at most one end of infinite measure. In the compact case,

X has no ends at all, so the original Oxtoby-Ulam Theorem is subsumed by this result (Corollary 38) for sigma-compact manifolds. It can be shown moreover (Theorem 43(iii)) that if h^* is componentwise weak mixing on the ends, then h can be approximated by homeomorphisms in $\mathcal{M}[X, \mu]$ possessing any property which is generic for transformations of an infinite Lebesgue space, that is, for $\mathcal{G}[X, \mu]$. Even when the manifold X does not have an end structure permitting this type of homeomorphism, $\mathcal{M}[X, \mu]$ always contains a homeomorphism possessing such a property (Corollary 44). Our measure theoretic results for homeomorphisms in $\mathcal{M}[X, \mu]$ sometimes have topological consequences. For example, we are able to apply our measure theoretic techniques to give short proofs of the existence of a fixed point (only one, unfortunately) for an area-preserving, orientation-preserving, homeomorphism of the torus with mean rotation zero (Conley-Zehnder-Franks Theorem) and for an area-preserving twist map of the annulus (Poincaré's Last Geometric Theorem). In this vein, we can also prove (Theorem 47) that $\mathcal{M}[X, \mu]$ always contains a homeomorphism which possesses *maximal chaos*, a property which gives chaos in the sense of Devaney with a maximal constant in the definition of sensitive dependence on initial conditions.

As this is a survey article, we will not present most of the proofs. In particular, the proofs of the main results are usually long and will only be outlined, if given at all. However, to give the reader a flavor of the way our techniques are used, we will give many of the shorter proofs, and those of all new results. The reader can find the missing details in the original papers or in [28].

Throughout this article the dimension n of the manifold X will always be assumed to be at least 2, and the measure μ on the manifold will always be assumed to be an OU measure (as defined in the next section). Sometimes we will repeat these assumptions for emphasis.

1.2 The spaces $\mathcal{M}$ and $\mathcal{G}$

In this section we define various topologies on two spaces, $\mathcal{M}$ and $\mathcal{G}$, of μ-preserving transformations of a manifold X. The manifold X is always assumed to be sigma-compact, that is, it can be represented as the union of a countable family of compact sets. We allow the possibility of a manifold boundary, which we denote by ∂X. The manifold X is metrizable and to simplify the statement of certain results we will further assume that the metric d has the property that if $d(x, y) < \delta$ then there is an arc from x to y of length less than δ. The measure μ will be assumed to be an *OU measure* (for Oxtoby-Ulam), which means it is a Borel measure on X with the following three properties:

1. It is nonatomic (zero for points)

2. It is locally positive (positive on nonempty open sets)

3. It is zero on the set of boundary points, $\mu(\partial X) = 0$.

Since a Borel measure is finite on compact sets, μ is a finite measure when X is compact and always a sigma-finite measure (X is a countable union of finite measure sets), since X is assumed to be sigma-compact. Observe that the three conditions which define an OU measure are topologically invariant. By this we mean that if μ has one of these properties then so does any measure 'homeomorphic to μ', that is, of the form $\nu(A) = \mu(h(A))$ for measurable sets A. We write this property as simply $\nu = \mu h$. Moreover, the following has been proved by von Neumann [65] and, independently by Oxtoby and Ulam [55].

Theorem 1 (Homeomorphic Measures Theorem) *Let ν_1 and ν_2 be two OU measures on a common compact manifold X with $\nu_1(X) = \nu_2(X)$. Then there is a self homeomorphism of X which fixes the boundary and satisfies $\nu_2 = \nu_1 h$.*

This result has been extended to the Hilbert cube by Oxtoby and the second author [57] and to noncompact manifolds (with additional conditions related to the ends of the manifold) by Berlanga and Epstein [30].

The paper is mainly about properties of the space $\mathcal{M}[X,\mu]$ consisting of all μ-preserving homeomorphisms of X. The main idea of the paper is to obtain information about $\mathcal{M}[X,\mu]$ by viewing it as a subset of the space $\mathcal{G} = \mathcal{G}[X,\mu]$ consisting of all bimeasurable μ-preserving bijections (called automorphisms) of X. The above remarks demonstrate that when X is compact the underlying measure space (X,Σ,μ) is finite, and for a general sigma-compact X it is sigma-finite. So while the nature of the space $\mathcal{M}[X,\mu]$ depends on both μ and X, the space $\mathcal{G}[X,\mu]$ is really one of only two types, depending on whether $\mu(X)$ is finite. That is, $\mathcal{G}$ is the space of all automorphisms of either a finite or infinite sigma-finite Lebesgue measure space. In keeping with our perspective of $\mathcal{M}$ as a subset of $\mathcal{G}$, we will define our topologies on the larger space $\mathcal{G}$ and obtain topologies on $\mathcal{M}$ by relativising. The reader should keep in mind that we view elements of $\mathcal{G}$ as being defined only up to sets of measure zero, while for homeomorphisms in $\mathcal{M}$ there is no such ambiguity.

The most important topology that has been introduced on $\mathcal{G}$ is the so called *weak topology*, under which a sequence $g_i \in \mathcal{G}$ converges to a limit g if $\mu(g_iB + gB) \to 0$ for all finite measure sets B. This topology is defined on $\mathcal{G}$ by the basic open neighborhoods

$$\mathcal{W}(g,\delta,B_1,\ldots,B_m) = \{f \in \mathcal{G} : \mu(fB_i + gB_i) < \delta, i = 1,\ldots,m\},$$

where δ is positive, $\mu(B_i) < \infty$, and '+' denotes symmetric difference. We note that the weak topology on $\mathcal{G}$ is topologically complete, that is, it is induced by a metric which is complete.

Roughly speaking, the space $\mathcal{G}$ is the subject of what is known as abstract or measure theoretic ergodic theory. However this paper assumes no knowledge of that theory. We will, however, need to know about three types of automorphisms in $\mathcal{G}$, defined by the successively stronger properties of μ-recurrence, ergodicity and weak mixing. An automorphism $g \in \mathcal{G}$ is called *μ-recurrent* if for any set A of positive measure, μ-almost all points of A have g-orbits which return to A. An automorphism g is called *ergodic* if every set which is g-invariant (this means $\mu(A + gA) = 0$) satisfies either $\mu(A) = 0$ or $\mu(X - A) = 0$. An automorphism

g is called *weak mixing* if the product automorphism $g \times g$ is ergodic on $X \times X$. The set of ergodic automorphisms and the set of weak mixing automorphisms in $\mathcal{G}$ are both known to be G_δ sets with respect to the weak topology [45].

The topology of interest on the space $\mathcal{M}[X,\mu]$ is the topology of uniform convergence on compact sets (called the *compact-open topology*). This reduces to the familiar uniform topology when X is compact. Since our constructions often will involve the uniform approximation of a homeomorphism $h \in \mathcal{M}[X,\mu]$ by a discontinuous automorphism $g \in \mathcal{G}[X,\mu] - \mathcal{M}[X,\mu]$, we will need to define these topologies on the larger space $\mathcal{G}$. So we define the compact-open topology on $\mathcal{G}[X,\mu]$ by specifying as basic open sets the neighborhoods of $g \in \mathcal{G}[X,\mu]$ given by

$$\mathcal{C}(g,K,\delta) = \{f \in \mathcal{G}[X,\mu] : d(f(x), g(x)) < \delta, \text{ for } \mu\text{-a.e. } x \in K\},$$

where K is a compact subset of the manifold X and δ is a positive number. We will sometimes use the simplifying notation $||f-g||_K = \text{ess sup}_{x \in K} d(f(x), g(x))$, with g taken to be the identity when it is not present, and K taken to be the whole space X when it is not present. Thus $||f||_K = \text{ess sup}_{x \in K} d(f(x), x)$.

The compact-open topology on $\mathcal{G}$ is finer than the weak topology [6]. The relative topology on $\mathcal{M}[X,\mu]$ is the usual compact-open topology (without the 'almost every' proviso), and is topologically complete. When K is compact, this is the uniform topology given by the uniform metric $||f-g||$. This fact will enable us to obtain existence results through the use of the Baire Category Theorem, which we state in the following form.

Theorem 2 *In a complete metric space, the intersection of countably many dense open sets is a dense set.*

A set which is the countable intersection of open sets is called a G_δ set, so we could equally have ended the statement of the theorem with 'a dense G_δ set'. We call a set *generic* if it contains a dense G_δ set. Such a set is large in the topological sense and may be said to represent 'the general case'. For another formulation of the notion 'prevalence' as it relates to ergodicity, see the problems posed by Mycielski in [54].

1.3 Known Genericity Results in $\mathcal{M}$,$\mathcal{G}$

In this section we review several earlier results which established that certain properties are generic in either $\mathcal{G}$ or $\mathcal{M}$. We begin with the results for compact manifolds and for finite measure spaces. The first result of the type we will consider is was established by Oxtoby and Ulam [55] (see [3], [4] for a combinatorial proof). Its historical significance was discussed at length in the Introduction.

Theorem 3 (Oxtoby and Ulam, 1941) *Let μ be an OU measure on a compact manifold X of dimension at least 2. Then the ergodic homeomorphisms constitute a dense G_δ subset of $\mathcal{M}[X,\mu]$ with respect to the uniform topology. In particular, every compact manifold supports an ergodic homeomorphism.*

A few years later, Halmos [43] established a similar result for the space $\mathcal{G}$.

Theorem 4 (Halmos, 1944) *Let $\mathcal{G} = \mathcal{G}[X, \mu]$ be the space of all automorphisms of a finite Lebesgue space $[X, \mu]$. Then the ergodic automorphisms of $[X, \mu]$ form a dense G_δ subset of $\mathcal{G}$, with respect to the weak topology.*

Halmos remarks on the similarity and differences between his result and the earlier result of Oxtoby and Ulam in his paper [43].

> The first theorem of this type is due to J. C. Oxtoby and S. M. Ulam ... Their topology is however, very different from mine and depends on the topological and metric (as opposed to purely measure theoretic) structure of the underlying space.

Later in the same paper (page 12) he states:

> ... there is, however, no implication between Theorem 6 [Theorem 3 above] and the corresponding result of Oxtoby and Ulam: they define a stronger topology and I consider a wider class of transformations.

The main thrust of this paper is to show that, Halmos' remark notwithstanding, there are indeed implications and relationships between genericity results in the spaces $\mathcal{G}$ and $\mathcal{M}$. These relationships are based on results established by the first author (Theorems 11, 21, 22, 23) subsequent to Halmos' remarks.

Proceeding chronologically, the next genericity result was established by Halmos [44] later in the same year. It is based on the following approximation theorem. (An automorphism is called *antiperiodic* if its periodic points constitute a set of measure zero.)

Theorem 5 (Halmos, 1944) *Let $\mathcal{G}$ be the space of all automorphisms of a finite Lebesgue space, and let $f \in \mathcal{G}$ be antiperiodic. The the conjugacy class of f, the set $\left\{g^{-1} f g : g \in \mathcal{G}\right\}$, is dense in $\mathcal{G}$ with respect to the weak topology.*

Since the weak mixing automorphisms form a conjugate invariant, G_δ subset of $\mathcal{G}$ and a particular weak mixing automorphism can be easily constructed, Halmos obtained the following specific consequence.

Theorem 6 (Halmos, 1944) *Let $\mathcal{G}$ denote the space of all automorphisms of a finite Lebesgue space. Then the weak mixing automorphisms form a dense G_δ subset of $\mathcal{G}$, with respect to the weak topology.*

A quarter century later, Katok and Stepin [47] established, as part of a wider study, a long sought analog of Halmos' result for measure preserving homeomorphisms. (A combinatorial proof of this result was later obtained by the first author [5].)

Theorem 7 (Katok and Stepin, 1970) *If μ is an OU measure on a compact manifold, then the weak mixing homeomorphisms constitute a dense G_δ subset of $\mathcal{M}[X,\mu]$ with respect to the uniform topology. In particular, every compact manifold supports a weak mixing homeomorphism.*

A superficial look at these four (and other) genericity results for compact manifolds or finite Lebesgue spaces suggests a connection between genericity in the spaces $\mathcal{G}$ and $\mathcal{M}$. Namely, the same properties seem to be generic in both spaces. Of course since the compact-open topology is finer than the weak topology it follows that if a property is G_δ in $\mathcal{G}$ with respect to the weak topology, then it is a G_δ subset of $\mathcal{M}$ (possibly empty though) with respect to the relative compact-open topology, which we call the uniform topology.

The connection later established by the first author between genericity in the two spaces can be stated as follows. It will be proved in section 2.6 as Theorem 23.

Theorem: *Let μ be an OU measure on a compact manifold X. Let $\mathcal{F}$ be a conjugate-invariant subset of $\mathcal{G}[X,\mu]$ which is dense and G_δ with respect to the weak topology. Then $\mathcal{F}\cap\mathcal{M}[X,\mu]$ is a dense G_δ subset of $\mathcal{M}[X,\mu]$ with respect to the compact open topology.*

We turn now to a discussion of prior results on genericity of properties for $\mathcal{G}$ and $\mathcal{M}$ when X is noncompact and $\mu(X)$ may be infinite. We know of no such results for homeomorphisms of noncompact manifolds. However the following result for automorphisms of infinite Lebesgue spaces was established in [63] and [36].

Theorem 8 (Sachdeva 71, Choksi and Kakutani 79) *The subset $\mathcal{E}$, consisting of ergodic automorphisms, is a dense G_δ subset of the space of all automorphisms $\mathcal{G}$ of an infinite sigma-finite Lebesgue space, with respect to the weak topology.*

2 Generic Dynamics on Compact Manifolds

2.1 The embedding $\mathcal{M}\subset\mathcal{G}$

In order to obtain uniform topology approximation results within the space $\mathcal{M}[X,\mu]$, we will often use an intermediate result which uniformly approximates a homeomorphism of the topological manifold X by a (discontinuous) automorphism of the underlying measure space $[X,\mu]$. That is, given an $h\in\mathcal{M}[X,\mu]$, it produces a $g\in\mathcal{G}[X,\mu]$ with some specified ergodic theoretic property $\mathcal{V}\subset\mathcal{G}[X,\mu]$, with $||h^{-1}g||<\varepsilon$. Suppose that $\mathcal{V}$ is open in the weak topology, and hence so is the set $\mathcal{U}=h^{-1}\mathcal{V}$. Thus the difference automorphism $f=h^{-1}g$ satisfies the two properties

1. $||f||<\varepsilon$, and
2. $f\in\mathcal{U}$, where $\mathcal{U}$ is open in the weak topology.

Suppose we could approximate the *automorphism* f by a *homeomorphism* $\hat{f} \in \mathcal{M}[X,\mu]$ which retained these two properties. Then the composition $h\hat{f}$ would belong to the specified set $h\mathcal{U} = \mathcal{V}$ and would be uniformly close to h. So this process would obtain the required uniform approximation *within* the space $\mathcal{M}[X,\mu]$.

The following result shows that we can indeed always approximate an automorphism by a homeomorphism in this manner.

Theorem 9 *Let μ be an OU measure on a compact manifold X. Let $g \in \mathcal{G}[X,\mu]$ satisfy $||g|| < \varepsilon$. Then given any weak topology open neighborhood $\mathcal{V}$ of g, there is a homeomorphism $h \in \mathcal{M}[X,\mu] \cap \mathcal{V}$ which satisfies $||h|| < \varepsilon$. Furthermore h can be chosen so as to fix the boundary of the manifold. In particular, $\mathcal{M}[X,\mu]$ is dense in $\mathcal{G}[X,\mu]$ with respect to the weak topology.*

This theorem has the following corollary which makes precise the motivating ideas given above the theorem.

Corollary 10 *Let $\mathcal{V}$ be a G_δ subset of $\mathcal{G}[X,\mu]$ with respect to the weak topology. If the uniform topology closure of $\mathcal{V}$ contains $\mathcal{M}[X,\mu]$, then $\mathcal{V} \cap \mathcal{M}[X,\mu]$ is a dense G_δ subset of $\mathcal{M}[X,\mu]$ with respect to the uniform topology.*

Proof. Write $\mathcal{V} = \cap_i \mathcal{V}_i$, where each of the countable number of sets $\mathcal{V}_i$ is open in the weak topology and has a uniform closure which contains $\mathcal{M}[X,\mu]$. By the argument given above the statement of Theorem 9 it follows that $\mathcal{V}_i \cap \mathcal{M}[X,\mu]$ is dense in $\mathcal{M}[X,\mu]$ with respect to the uniform topology. (That is, every $h \in \mathcal{M}[X,\mu]$ can be uniformly approximated by a homeomorphism $h\hat{f} \in \mathcal{M}[X,\mu]$ which belongs to $\mathcal{V}$.) The fact that $\mathcal{V} \cap \mathcal{M}[X,\mu]$ is dense in the relative uniform topology on $\mathcal{M}[X,\mu]$ is a consequence of the fact that the uniform topology is finer than the weak topology. Hence the Baire Category Theorem, Theorem 2, asserts that the countable intersection

$$\mathcal{V} \cap \mathcal{M}[X,\mu] = \cap_i (\mathcal{V}_i \cap \mathcal{M}[X,\mu])$$

of the dense open sets on the right hand side is a dense G_δ subset $\mathcal{M}[X,\mu]$. ■

Actually there is a stronger form of Theorem 9 in which the weak topology on $\mathcal{G}[X,\mu]$ is replaced by the finer one given by the metric $\mu(\{x : f(x) \neq g(x)\})$. The weaker form given above will be sufficient for our purposes, but for completeness we state as follows the stronger form established by the first author in [6].

Theorem 11 (Lusin Theorem for Measure Preserving Homeomorphisms) *Let μ be an OU measure on a compact manifold X. Let $g \in \mathcal{G}[X,\mu]$ satisfy $||g|| < \varepsilon$. Then given any $\delta > 0$, there is a homeomorphism $h \in \mathcal{M}[X,\mu]$ with $||h|| < \varepsilon$ such that $\mu(\{x \in X : h(x) \neq g(x)\}) < \delta$. Furthermore h can be chosen so as to fix the boundary of the manifold.*

The role of this result is similar to that of Lusin's Theorem in real variable theory. Versions of this result without the condition on the uniform norm have been proved by Oxtoby [56], White [66], and the first author and Edwards [16].

2.2 Short Proof of the Oxtoby-Ulam Theorem

In this subsection we use the Lusin Theorem, Theorem 9 given above, to give a new short proof of the result of Oxtoby and Ulam (Theorem 3), that ergodicity is generic for measure preserving homeomorphisms of compact manifolds. For simplicity, we limit the presentation to the case where X is the unit cube I^n, $n \geq 2$, and μ is Lebesgue measure λ. The same proof applies to the general case stated as Theorem 3, since any compact manifold with an OU probability measure can be represented as the unit cube with suitable boundary identifications. (See [28, Theorem 9.6], which is a measure preserving extension of a result of Brown [34].)

The advantage of carrying out the proof on the cube is that we make use of the very natural dyadic decompositions. A *dyadic cube* of order m is the n-fold product of closed intervals of the form $[k/2^m, (k+1)/2^m]$, and a *dyadic permutation* of order m is an automorphism $P \in \mathcal{G}[I^n, \lambda]$ which permutes these cubes by simple rigid translation. (Since P is viewed simply as an automorphism of the cube, where zero measure behavior is irrelevant, we do not need to clarify the ambiguity of the action of P on the boundary of the dyadic cubes.)

The following result of Lax [49], with cyclicity added by the author in various ways ([3],[4],[5]) says that the uniform topology closure of the set of cyclic dyadic permutations contains $\mathcal{M}[X, \mu]$.

Theorem 12 (Lax) *Given $h \in \mathcal{M}[I^n, \lambda]$ and $\varepsilon > 0$ there exists a cyclic dyadic permutation P, of arbitrarily high order, such that $||P - h|| < \varepsilon$.*

Proof. The idea of the proof is as follows. First choose a dyadic decomposition into dyadic cubes $D_1, \ldots, D_N$, $N = 2^{mn}$, of sufficiently high order m (which will depend on ε and the uniform modulus of continuity of h). Then define a cube D_i to 'know' a cube D_j if $h(D_i) \cap D_j \neq \emptyset$. Since h is volume preserving, any set of k cubes together know at least k cubes. So by the Marriage Theorem there is a permutation Q of the cubes that assigns each cube to one that it knows. This process ensures that Q is uniformly close to h. There are several ways to uniformly approximate a dyadic permutation by a cyclic one. The easiest way is to number the cubes so that D_i and D_{i+1} share a common $n-1$ dimensional face, and to use the following lemma from [5]: given any permutation σ of the set $J = \{1, \ldots, N\}$, there is a cyclic permutation $\hat{\sigma}$ of J such that $|\hat{\sigma}(i) - \sigma(i)| \leq 2$, for all $i \in J$. Then if $Q(D_i) = D_{\sigma(i)}$, define $P(D_i) = D_{\hat{\sigma}(i)}$. ■

For the sake of completeness, we mention the most general version of the fact that any permutation can be well uniformly approximated by a cyclic one. Of course for discrete, equiprobable measure spaces, this can be thought of as a uniform approximation of an arbitrary automorphism of the space by an ergodic (cyclic) one. This result is taken from [7].

Theorem 13 *Let $G = (X, A)$ be a finite connected graph with nodes X and arcs A. Then given any permutation $P : X \to X$, there is a cyclic permutation $Q : X \to X$ such that for any node $x \in X$ the distance between $P(x)$ and $Q(x)$ is at most 6 arcs.*

We now show how the Measure Preserving Lusin Theorem (Theorem 9) can be used to give a short proof of the Oxtoby-Ulam Theorem (Theorem 3). This proof is previously unpublished, although a similar version will appear in our forthcoming book [28].

Proof of Theorem 3 for $X = I^n$, $\mu = \lambda$ (Lebesgue measure) . Since the subset $\mathcal{E}$ of $\mathcal{G}[I^n, \lambda]$ consisting of ergodic automorphisms is G_δ in the weak topology (see [45]), Corollary 10 reduces the problem to showing the following: *given any $h \in \mathcal{M}[I^n, \lambda]$ and any $\varepsilon > 0$, there is an ergodic automorphism $f \in \mathcal{G}[I^n, \lambda]$ with $||h - f|| < \varepsilon$.*

To demonstrate the italicized claim, apply Lax's Theorem above to obtain a cyclic dyadic permutation $P \in \mathcal{G}[I^n, \lambda]$ of some dyadic decomposition D_i, $i = 1, \ldots, N$, with

$$||h - P|| + diameter(D_i) < \varepsilon.$$

Number the cubes D_i so that $P(D_i) = D_{i+1}$ for $i < N$, and $P(D_N) = D_1$. Let $e : D_1 \to D_1$ be any ergodic automorphism of this dyadic cube (it is easy to explicitly define such an e) and extend it to an automorphism of the whole cube I^n (with the same name) by setting it equal to the identity off of D_1. Define $f = eP$. Observe that under f all points in any dyadic cube D_i, $i < N$, move to the equivalent point (under translation) in the cube D_{i+1}. Points in D_N move to the equivalent point in D_1 and then to the e-image of that point, also in D_1. We claim that the automorphism e is ergodic. For those readers familiar with skyscraper constructions, this can be seen as a simple application of the fact (due to Kakutani) that a skyscraper built on an ergodic base transformation is ergodic. We give the following direct proof. Suppose that f is not ergodic, so that by definition there is an f-invariant set S with $0 < \lambda(S) < 1$. Then the sets $S_i = S \cap D_i$ also have intermediate volume $(0 < \lambda(S)/N < 1/N)$. It follows from the definitions of f and the S_i that $f^N(S_i) = S_i$. But $f^N(S_1) = e(S_1)$ which cannot equal S_1 because e was assumed to be ergodic. This contradiction means that our assumption that f was not ergodic was false. Thus f is an ergodic automorphism of the cube I^n with

$$||h - f|| \leq ||h - P|| + ||P - f|| \leq \varepsilon - diameter(D_i) + diameter(D_i) = \varepsilon.$$

■

2.3 Approximation by Dyadic Permutations

To give another illustration of the uses of the Measure Preserving Lusin Theorem, we apply it to prove the following important early result of Halmos [43] as a corollary of Lax's Theorem. The proof given below has been presented in talks by the first author, but has not been previously published.

Corollary 14 *Let $g \in \mathcal{G}[I^n, \lambda]$, $n \geq 2$, be an arbitrary automorphism. Then any weak topology neighborhood $\mathcal{W}$ of g contains a cyclic dyadic permutation of arbitrarily high order.*

Proof. By the Measure Preserving Lusin Theorem (actually we need only the weaker form stated as Theorem 9 - and without any norm bound), the weak topology open set $\mathcal{W}$ contains a homeomorphism $h \in \mathcal{M}[I^n, \lambda]$. Since the uniform topology on $\mathcal{G}[I^n, \lambda]$ is finer than the weak topology, there is a uniform topology neighborhood $\mathcal{C}$ of h with $h \in \mathcal{C} \subset \mathcal{W}$. By Lax's Theorem there is a cyclic dyadic permutation of arbitrarily high order belonging to the uniform neighborhood $\mathcal{C}$ of h, and hence to the larger set $\mathcal{W}$, as required. ■

We observe that since there is a measure preserving bijection between the line segment I^1 and the unit square I^2 which takes dyadic intervals of even order into dyadic squares, the above corollary applies as well to the case $n = 1$, which was the setting for Halmos' original result.

In his original paper, Halmos says that the above approximation result

> ... has the rank of a 'folk theorem' .... It has a satisfying intuitive content: it says that, in the limit, every measure preserving transformation is obtained by cutting up the space (with an ordinary pair of Euclidean scissors) into a finite number of pieces and then merely permuting the pieces.

We now consider whether there is a version of Halmos' result for measure preserving homeomorphisms, with the approximation in the uniform topology. Of course a permutation of all the dydadic cubes D_i, $i = 1, \ldots, 2^{mn}$, of any order m is not continuous (except for the identity permutation). However if only the subcubes D_i^α concentric to D_i and with a fractional side length $\alpha < 1$ are rigidly permuted, this involves no topological contradiction. Such a permutation of small cubes could be extended to a homeomorphism of the full cube I^n. We call the cubes D_i^α, *α-cubes of order m*. We then continue to gradually define a transformation on the rest of the cube I^n by rigidly permuting the α-cubes of some finer dyadic subdivision which are not contained in any of the previously permuted cubes. In this manner we can define a transformation on all of I^n. The surprising fact is that this can be accomplished by a homeomorphism. This result can be phrased on an arbitrary compact manifold, with cyclically permuted families of topological n-balls. However the sharper form of the result (with rigid motions) is best formulated for the cube, possibly with boundary identifications. The following is taken from [3] and can be found with a full proof in [14].

Theorem 15 *Let X be a compact manifold represented as the unit cube I^n, $n \geq 2$, with possible boundary identifications, and let λ denote Lebesgue measure. Let α, $0 \leq \alpha < 1$, be any dyadic rational number (of the form $j/2^i$) and let ε be any positive number. Given any homeomorphism $h \in \mathcal{M}[X, \lambda]$, there exists another homeomorphism $f \in \mathcal{M}[X, \lambda]$ with $||f|| < \varepsilon$ and equal to the identity on ∂I^n, and an increasing sequence of positive integers $m_1, m_2, \ldots,$ with the following property: For each $k = 1, 2, \ldots,$ the α-cubes of order m_k which are not contained in any of the α-cubes of orders $m_1, m_2, \ldots, m_{k-1}$ are rigidly permuted in a single cycle by the homeomorphism fh.*

The proof is the same for all α, but is only carried out for $\alpha = 1/2$ in [3]. The proof relies on the Annulus Theorem, and so some dimensions were excluded from the original proof. The Annulus Theorem now has no missing dimensions (see [48] and [61]). The above result generalizes a similar result of Oxtoby and Ulam [55] which applied only to the unit cube (without boundary identifications), or more generally to n-manifolds embedded in R^n, and only to homeomorphisms h equal to (or isotopic to) the identity on the boundary of the cube. The interpretation of this result varies, depending on whether the fraction α is positive or zero. In the former case the total volume of all the permuted α-cubes is the full volume of the cube (that is, 1), so that the uniform approximation fh is measure theoretically periodic. Thus homeomorphisms with almost all points periodic are dense in the space $\mathcal{M}[X,\mu]$, with respect to the uniform topology. For the case of the cube we can state moreover the following.

Corollary 16 *Every volume preserving homeomorphism of the unit cube I^n, $n \geq 2$, can be uniformly approximated by another which is measure theoretically periodic and is locally linear almost everywhere.*

In the case that $\alpha = 0$ the α-cubes are simply the points at the centers of the larger cubes, and the permuted cubes (points) have measure zero. So there are no measure theoretical consequences. However for any open subsets U and V of the cube there will be a periodic fh-orbit of 0-cubes (dyadic centers) which enters both U and V. This property implies that the uniform approximation fh is chaotic in the sense of Devaney, a term that we will define later in section 3.5, where we prove a stronger result (Theorem 46) valid for all sigma-finite manifolds. In any case, Theorem 15 gives the following.

Corollary 17 *Every volume preserving homeomorphism of the unit cube I^n, $n \geq 2$, can be uniformly approximated by another one which is chaotic in the sense of Devaney.*

Actually a stronger result, with density replaced by genericity, has been proved by Daalderop and Fokkink [38], using a version of the above corollary due to Aarts and Daalderop [2].

2.4 Multiple Tower Rokhlin Theorem

In his proof that weak mixing automorphisms are generic among automorphisms of a finite Lebesgue space, Halmos proved a result (due also to Rokhlin and Kakutani) which says that an antiperiodic automorphism has a large tower. A g-tower of height m for an automorphism $g \in \mathcal{G}$ consists of disjoint iterates $B, gB, \ldots, g^{m-1}B$ of a base set B. Rokhlin's lemma says that if $g \in \mathcal{G}[X,\mu]$ is antiperiodic then for any integer m and positive ε we can find a g-tower of height m with $\mu\left(X - \bigcup_{i=0}^{m-1} g^i B\right) < \varepsilon$. We note for future reference that we could make the last inequality an equality by shaving off a piece of B. Note that since any set (in particular the set $X - \bigcup_{i=0}^{m-1} g^i B$) can be described as a tower of height 1, we may state this result ([62],[46], [44], [45]) as follows:

Theorem 18 (Rokhlin Lemma) *Let m be a positive integer and let $\varepsilon < 1$ be a positive number. Any antiperiodic automorphism of a Lebesgue probability space can be represented as the disjoint union of a tower of height m and a tower of height 1, with respective measures $1-\varepsilon$ and ε.*

We will need the following stronger version of Rokhlin's Theorem which states that we can prescribe in advance any sequence of towers, as long as their heights are (like the numbers m and 1) relatively prime.

Theorem 19 (Multiple Tower Rokhlin Theorem) *Let $g \in \mathcal{G}[X,\mu]$ be any antiperiodic automorphism of a Lebesgue probability space $[X,\Sigma,\mu]$. Then given any countable probability distribution $\pi = (\pi_1, \pi_2, \dots)$ such that the k's with $\pi_k > 0$ are relatively prime, there is a partition of X into disjoint towers $B_k \cup gB_k \cup \cdots \cup g^{k-1}B_k$, $k = 1, 2, \dots$, with $\mu\left(B_k \cup gB_k \cup \cdots \cup g^{k-1}B_k\right) = \pi_k$.*

This result was proved for finite distributions in [9] and subsequently extended to the countable case in [11]. A generalization to the nonsingular case was established in [20]. Alternative proofs were given in [39] and [60] for the finite and countable versions, respectively.

2.5 Denseness of Conjugacy Classes in $\mathcal{G}$

Given any automorphism $g \in \mathcal{G} = \mathcal{G}[X,\mu]$, we define its *conjugacy class* in $\mathcal{G}$ to be the set $\mathcal{C}_{\mathcal{G}}(g) = \left\{f^{-1}gf : f \in \mathcal{G}\right\}$. To prove that the weak mixing automorphisms are dense in $\mathcal{G}$ with respect to the weak topology, Halmos [44] first established what we have called Theorem 5, and which we can rephrase as follows:

If $g \in \mathcal{G}[X,\mu]$ is an antiperiodic automorphism of a Lebesgue probability space, then its conjugacy class $\mathcal{C}_{\mathcal{G}}(g)$ is dense in $\mathcal{G}$ with respect to the weak topology.

It was suggested by Katok and Stepin [47] that an analogous result for the space $\mathcal{M}[X,\mu]$ might be used to establish genericity results for measure preserving homeomorphisms of a compact manifold X. While we cannot prove that version, we will show in this section (Theorem 21) that such a result is indeed true in $\mathcal{G}[X,\mu]$ with the uniform topology. The significance for the subspace $\mathcal{M}$ will be shown in the next section, using the Lusin Theorem for Measure Preserving Homeomorphisms, in the weaker form of Theorem 9.

We first show how the Multiple Tower Rokhlin Theorem, Theorem 19, can be used to establish a stronger but still purely measure theoretic version of Halmos' result (Theorem 5). We seek conditions on $h \in \mathcal{G}[X,\mu]$ and a measurable set F such that some conjugate $\bar{g} = f^{-1}gf$ of the given antiperiodic automorphism g equals h pointwise on the set F. Observe that if F had a nontrivial h-invariant subset, then it would also be $\bar{g}$ and g invariant, which would not be possible if g were ergodic. Furthermore if F had a subset F_0 of measure $1/2$ with

$h(F_0) \cup F_0 = X$ then the same would be true of g, which would not be possible if g were weak mixing. The condition that h has such a periodic set (possibly with a larger period m than 2) is the same as saying that h^m is not ergodic. So this problem can be avoided by saying that h^m is ergodic for all $m \geq 1$, a condition called *totally ergodic.* The following result [11] says that if these problems are excluded, a conjugate equal to the target automorphism h on the given set F can be obtained.

Theorem 20 *Let $g, h \in \mathcal{G}[X, \mu]$ be given, with g antiperiodic. Let F be a measurable set with $\mu(F) < 1 = \mu(X)$. Assume either*

(i) *h is ergodic and $\mu(F \cup hF) < 1$, or*

(ii) *h is totally ergodic.*

Then for some $f \in \mathcal{G}[X, \mu]$ the conjugate $\bar{g} = f^{-1} g f$ satisfies

$$\bar{g}(x) = h(x) \text{ for } \mu\text{-a.e. } x \in F.$$

Proof. For $k = 2, 3, \ldots$, consider the set $A_k \subset F - hF$ consisting of all points of F which leave F on the $k-1$'th iterate of h. Define $A_1 = X - (F \cup hF) = T_1$. Then the set $T_k = A_k \cup h\, A_k \cup \cdots \cup h^{k-1} A_k$ is an h-tower of some height k and measure π_k, $F \cup hF = \cup_{k=2}^{\infty} T_k$, and since h is ergodic we have $X = \cup_{k=1}^{\infty} T_k$. Under condition (i), we have $\pi_1 = \mu(A_1) > 0$. Suppose condition (i) does not hold but condition (ii) holds. Then if the k's for which $\pi_k > 0$ are not relatively prime and their greatest common divisor is p, the set $P = \cup_{k \geq 2} \cup_{0 \leq ip \leq k-1} h^{ip}(A_k)$ has h-period p. This implies that h^p is not ergodic and hence, contrary to hypothesis, that h is not totally ergodic. Thus in either case the k's for which $\pi_k > 0$ are relatively prime and so we can apply the Multiple Tower Rokhlin Theorem (Theorem 19) to obtain a partition of X into g-towers $B_k \cup gB_k \cup \cdots \cup g^{k-1} B_k$ of height k and measure π_k. We can define the required automorphism f on $F \cup hF$ by setting $f(A_k) = B_k$ and then we extend it in the natural way to the rest of X. (The details omitted here can be found in [11].) ∎

The following strengthening of Halmos' conjugacy approximation (Theorem 5), with the conjugacy class dense in the finer uniform topology, can be proved as a corollary of Theorem 19. In fact it was proved in the same paper [9] that established the *finite tower* version of Theorem 19 and hence does not actually require the full strength of that result. An extended version of this result is needed for noncompact manifolds, and is presented later as Corollary 42.

Theorem 21 *If $g \in \mathcal{G}[X, \mu]$ is an antiperiodic automorphism of a Lebesgue probability space arising from an OU measure μ on a compact manifold X, then its conjugacy class $\mathcal{C}_{\mathcal{G}}(g)$ is dense in $\mathcal{G}$ with respect to the uniform topology.*

Proof. We shall only prove here the fact that if X is the unit cube then $\mathcal{C}_{\mathcal{G}}(g)$ is dense in $\mathcal{M}[X, \mu]$. The full proof can be found in [9] or [11], where it is proved as a corollary of Theorem 19. The shorter but more restricted proof given here is instead based on Theorem 20. The restriction here to the approximation only of homeomorphisms in $\mathcal{M}[X,\mu]$ is not serious, as this version of the above theorem is in fact sufficient to prove the main results of the next section. Thus this proof could be used to prove the results of the next section for the special case where X is the unit cube, as was done in [8].

We will show here that given any homeomorphism $h \in \mathcal{M}[X,\mu]$ and any $\varepsilon > 0$, there is an automorphism $f \in \mathcal{G}[X,\mu]$ such that $\left|f^{-1}gf(x) - h(x)\right| < \varepsilon$ for μ-*a.e.* $x \in X$. We may assume that h is ergodic, as Theorem 3 (Oxtoby-Ulam) showed that the ergodic homeomorphisms are uniformly dense in $\mathcal{M}[X,\mu]$. [Note: the full proof of this Theorem does not require that h be ergodic, nor indeed does it require any result regarding the space $\mathcal{M}[X,\mu]$.] By Brouwer's Theorem, there is a fixed point $p = h(p)$. Let B be the open $\varepsilon/2$-ball centered at p. Define $F = h^{-1}(I^n - B)$. The closed set $F \cup h(F)$ does not contain p and consequently has volume less than 1. According to part (i) of the previous theorem, there is an automorphism $f \in \mathcal{G}[X,\mu]$ such that the conjugate $\bar{g} = f^{-1}gf$ satisfies $\bar{g}(x) = h(x)$ for μ-a.e. $x \in F$. For points $y \notin F$, both $\bar{g}(y)$ and $h(y)$ belong to the set B, which has diameter ε. ■

A stronger form of this result is presented later as Corollary 42.

2.6 Genericity results in $\mathcal{G}$ and $\mathcal{M}$

In this section we relate the genericity results stated in section 1.3 for $\mathcal{G}$ to those for $\mathcal{M}$. In particular, we show that any measure theoretic property which is generic in $\mathcal{G}$ is also generic in $\mathcal{M}$. The results in this section were obtained first [8] using a technique that applied only to manifolds with the fixed point property and subsequently generalized in [9] to arbitrary compact manifolds.

We begin with a result which treats the two spaces $\mathcal{G}$ and $\mathcal{M}$ in a similar fashion. Of course part (i) is not new and is given only for the sake of symmetry between the two spaces. The main result is part (ii). (A subset $\mathcal{S}$ of $\mathcal{G}[X,\mu]$ is called *conjugate invariant* if $\mathcal{S} = g^{-1}\mathcal{S}g$ for any $g \in \mathcal{G}[X,\mu]$.)

Theorem 22 *Let μ be an OU measure on a compact n-manifold X, $n \geq 2$. Let $\mathcal{F} \subset \mathcal{G}[X,\mu]$ be any weak topology G_δ subset which is conjugate-invariant and contains an antiperiodic automorphism. Then*

(i) *$\mathcal{F}$ is a dense G_δ subset of $\mathcal{G}[X,\mu]$ with respect to the weak toplogy, and*

(ii) *$\mathcal{F} \cap \mathcal{M}[X,\mu]$ is a dense G_δ subset of $\mathcal{M}[X,\mu]$ with respect to the uniform topology.*

Proof. If $\mathcal{F}$ is conjugate invariant and contains an antiperiodic automorphism a, then it contains its conjugacy class $\mathcal{C}_{\mathcal{G}}(a)$, which by Theorem 21 is dense in $\mathcal{G}[X,\mu]$ with respect to the uniform topology. Consequently $\mathcal{F}$ is a weak topology G_δ subset of $\mathcal{G}[X,\mu]$ which is dense in $\mathcal{G}[X,\mu]$ (and hence also in $\mathcal{M}[X,\mu]$) with respect to the uniform topology. So by Corollary 10 the set $\mathcal{F} \cap \mathcal{M}[X,\mu]$ is a dense G_δ subset of $\mathcal{M}[X,\mu]$ with respect to the uniform topology. This proves part (ii). To establish part (i) from part (ii) it is enough to observe that the uniform topology is finer than the weak topology (both considered on $\mathcal{G}[X,\mu]$) and that $\mathcal{M}[X,\mu]$ is dense in $\mathcal{G}[X,\mu]$ with respect to the weak topology (Theorem 9). Since it is assumed that $\mathcal{F}$ is G_δ in the weak topology this completes the proof of (i) – which is of course a result of Halmos which follows alternatively simply from his weak topology conjugacy approximation result, Theorem 5. ■

We conclude this section with an alternative statement of the above result which precisely expresses our claim that 'any property which is typical for automorphisms of a finite Lebesgue space is typical for homeomorphisms of a manifold preserving a fixed measure'.

Theorem 23 *Let μ be an OU measure on a compact n-manifold X, $n \geq 2$. Let $\mathcal{F}$ be any conjugate invariant subset of $\mathcal{G}[X,\mu]$ which is G_δ with respect to the weak topology. Then $\mathcal{F} \cap \mathcal{M}[X,\mu]$ is a dense G_δ subset of $\mathcal{M}[X,\mu]$ with respect to the uniform topology.*

Proof. Since the set $\mathcal{E} \subset \mathcal{G}[X,\mu]$ consisting of all ergodic automorphisms is a dense G_δ subset with respect to the weak topology (Halmos' 1944 result, our Theorem 4), it follows from the hypothesis on $\mathcal{F}$ and the Baire Category Theorem that also $\mathcal{E} \cap \mathcal{F}$ is a dense G_δ subset of $\mathcal{G}[X,\mu]$ with respect to the weak topology. Consequently $\mathcal{F}$ contains an ergodic (and hence antiperiodic) automorphism, so that the result to be proved follows from part (ii) of the previous theorem. ■

In particular, this result establishes the *existence* of μ-preserving homeomorphisms of any compact manifold which possess any given property which is typical for automorphisms of a finite Lebesgue space. For some manifolds, stronger existence results may be obtained. The most important example of this kind is the deep result of D. Lind and J.-P. Thouvenot [52] which established the existence in $\mathcal{M}\left[T^2,\lambda\right]$ (λ =area-preserving homeomorphisms of the torus T^2) of every finite entropy ergodic transformation. However, as their result does not obtain existence via Baire Category methods, it is not an extension of the Oxtoby-Ulam Theorem and does not belong to the primary subject of this article.

3 Generic Dynamics on Noncompact Manifolds

In this part of the survey we discuss extensions, and obstructions to extensions, of the Oxtoby-Ulam Theorem to noncompact manifolds. This area of research was initiated by the second author in 1979, when he proved that ergodicity is generic for volume preserving homeomorphisms of Euclidean n-space. The theory subsequently developed by the authors for noncompact manifolds differs substantially from that for compact manifolds (summarized for example in Theorem 23) in two respects. The first difference is topological: For compact manifolds, the manifold topology did not interact with the ergodic theory, that is, Theorem 23 applied equally well to all compact manifolds. On the contrary, the generic dynamics on a noncompact manifold depends crucially on its *end* structure - for example ergodicity is generic for volume preserving homeomorphisms on the one-end manifold R^2 but not on the two-end manifold $R \times [0,1]$. The second difference is measure-theoretical: The genericity theory for compact manifolds applies equally to all measure-theoretic properties. For noncompact manifolds, there are separate techniques and theories required, for example, to consider genericity of ergodicity compared with genericity of weak mixing.

The results in the remainder of the article are concerned with measured manifolds (X, μ) where X is sigma compact and μ is an OU measure. Although the case of a compact manifold is subsumed in this material, we have no further results for that case. So the reader should mainly be thinking of noncompact sigma-compact manifolds. Since X need not be compact, the OU measure μ need no longer be finite. The main thrust of this part of the survey relates to the case where μ is infinite - the case of a finite measure on a noncompact space is very similar to the compact case already considered. For example Theorem 43(i), on generic properties of homeomorphisms of a noncompact manifold preserving a *finite* measure, is essentially the same as Theorem 23 stated above for compact manifolds). So in particular, the space $\mathcal{G}[X, \mu]$ is now generally the space of all automorphisms of an *infinite* sigma-finite Lebesgue measure space, and it is to this measure theoretic structure that we relate our dynamics on manifolds. To understand our work, the reader will not need any background in the area of infinite measure ergodic theory, but if interested the recent book of Aaronson [1] is recommended.

We conclude this introduction to dynamics on noncompact manifolds with a brief comment on our use of the compact-open topology rather than the simpler uniform topology. (Of course these coincide for compact manifolds.) Consider the unit translation r, given by $(x, y) \to (x + 1, y)$ on the plane. It is not ergodic, nor even recurrent. Furthermore if another area preserving plane homeomorphism h is uniformly close to r, say $||h - r|| < 1/2$, then the same observations apply to h. However as the second author's result on generic ergodicity in R^2 shows, the use of the compact-open topology allows us to approximate r as close as we like on as large a compact set as we like, by an ergodic homeomorphism.

3.1 Generic dynamics on R^n

Although Oxtoby and Ulam did not consider in [55] (or elsewhere) the question of ergodic theoretic properties of volume preserving homeomorphisms of Euclidean space R^n, Ulam had earlier posed the following simpler question:

> Does there exist a homeomorphism h of the Euclidean space R^n with the following property? There exists a point p for which the sequence of points $h^n(p)$ is everywhere dense in the whole space.

This quotation is taken from Problem 115 of the famous *Scottish Book* [53] of 1935, a record kept by Polish mathematicians of their discussions at the 'Scottish Cafe' in Lwów, Poland. A transformation with the specified property is called *transitive*. Shortly thereafter, Besicovitch [31] gave an explicit example of a transitive homeomorphism of the plane. However it was not until 1979 that the existence of ergodic homeomorphisms of all spaces R^n (for $n \geq 2$ but $n \neq 4$) was established by Prasad [59], who proved moreover that ergodicity is generic in $\mathcal{M}[R^n, \lambda]$, where λ as usual denotes volume measure. The reason for the restriction $n \neq 4$ was that, at the time, the Annulus Theorem was not proved for that dimension. (With that gap filled by Quinn in 1982 [61], Prasad's original proof applies to all dimensions $n \geq 2$.) Subsequent to Prasad's result, Alpern [10][12] gave two proofs (the latter not requiring the Annulus Theorem and hence with no dimensional restriction) that any measure theoretic property typical for automorphisms of an infinite sigma-finite Lebesgue measure space (with respect to the weak topology) is typical for volume preserving homeomorphisms of R^n. We present the second of these here. The proof requires the following infinite measure analog of Theorem 20, which is due to J. Choksi and the authors [15].

Theorem 24 *Let $\mathcal{G}[X, \mu]$ denote the group of all automorphisms of the infinite sigma-finite nonatomic Lebesgue space $[X, \Sigma, \mu]$. Let h and g be two ergodic automorphisms in $\mathcal{G}[X, \mu]$ and let F be any measurable subset of X such that $\mu(X - (F \cup h(F))) = \infty$. Then there is an automorphism $f \in \mathcal{G}[X, \mu]$ such that the conjugate $\bar{g} = f^{-1}gf$ of g satisfies*

$$\bar{g}(x) = h(x) \text{ for } \mu\text{-a.e. } x \in F.$$

In fact the weaker version of the theorem, with the stronger hypothesis that $\mu(F) < \infty$, is all that is needed for the application given below in this section, and that version was already established by Choksi and Kakutani in 1979 [36]. We stated Theorem 24 in the stronger form above because this form will be required later in the proof of Theorem 43 (iii).

Theorem 25 *Let $\mathcal{F}$ be any conjugate invariant subset of $\mathcal{G}[R^n, \lambda]$ which is G_δ with respect to the weak topology. Then $\mathcal{F} \cap \mathcal{M}[R^n, \lambda]$ is a dense G_δ subset of $\mathcal{M}[R^n, \lambda]$ with respect to the compact-open topology.*

Proof. The subset $\mathcal{E}$ of $\mathcal{G}[R^n, \lambda]$ consisting of ergodic automorphisms is known [63][36] to be a dense G_δ subset, with respect to the weak topology. So the same argument used at the beginning of the proof of Theorem 23 shows that for some ergodic automorphism g, the full conjugacy class $\mathcal{C}_{\mathcal{G}}(g)$ belongs to $\mathcal{F}$.

By appealing to the Baire Category Theorem and taking intersections, it is enough to show that if $\mathcal{F}$ is additionally *open* in the weak topology, then $\mathcal{F} \cap \mathcal{M}[R^n, \lambda]$ is dense in $\mathcal{M}[R^n, \lambda]$ with respect to the compact-open topology. This in turn means that we need to show that the set

$$\mathcal{F} \cap \mathcal{M}[R^n, \lambda] \cap \mathcal{C}(h, K, \varepsilon) \neq \emptyset,$$

for any compact-open basic neighborhood $\mathcal{C}(h, K, \varepsilon)$. To simplify the proof, we use the second author's result on genericity of ergodic homeomorphisms in $\mathcal{M}[R^n, \lambda]$ to assume furthermore that the homeomorphism h is ergodic. (A proof can be carried out without using that result (see [28]), but it is more complicated.) In any case we may assume that the compact set K is a closed n-cube.

We now apply the previous result to the ergodic automorphisms h and g, with $F = K$, to obtain a conjugate $\bar{g}$ of g which equals h almost everywhere on K and belongs to the weak topology open set $\mathcal{F}$. Observe that for any $\delta > 0$, the automorphism $\tilde{g}$ defined by $\tilde{g} = h^{-1}\bar{g}$ satisfies

$$\tilde{g}\,(K) = K, ||\tilde{g}||_K < \delta, \text{ and } \tilde{g} \in h^{-1}\mathcal{F} \ .$$

Let K' be a large cube which contains K in its interior, and let $\breve{g} \in \mathcal{G}\,[R^n, \lambda]$ be any automorphism which leaves K' invariant and agrees with $\tilde{g}$ pointwise on K. If the set K' is chosen large enough, then $\breve{g}$ will be a good enough weak topology approximation to $\tilde{g}$ so that $\breve{g}$ will also belong to the weak topology open set $h^{-1}\mathcal{F}$.

We now apply the Lusin Theorem for Measure Preserving Homeomorphisms, in the weak form given as Theorem 9, to the automorphism $\breve{g}$, first on the invariant set K (with the norm bound δ) and then on the annulus $K' - K$ (without any norm bound). This gives us a homeomorphism $h_1 \in \mathcal{M}\,[R^n, \lambda]$ which is the identity on the boundary of each of the cubes K and K', and which approximates both $\breve{g}$ and $\bar{g}$ in the sense that

$$h_1\,(K) = K, ||\tilde{g}||_K < \delta, \text{ and } h_1 \in h^{-1}\mathcal{F}.$$

It follows that for δ sufficiently small, the homeomorphism hh_1 belongs to the set $\mathcal{F} \cap \mathcal{M}\,[R^n, \lambda] \cap \mathcal{C}\,(h, K, \varepsilon)$, which consequently cannot be empty. ■

3.2 The Ends of a Manifold

The reader may think that, just as Theorem 23 applied to *all* compact manifolds, Theorem 25 for Euclidean space applies equally to *all* sigma-compact manifolds. However this is not the case, as we shall demonstrate in Theorems 31 and 33 in section 3.3. In order to distinguish manifolds X where results like Theorems 23 and 25 hold from those where they fail, we must look at the ends of X and the interaction of the ends and the measure μ. In this section and the next, we define most of the concepts related to ends that we will subsequently need. Some of the definitions require some justification, and this can be found in [19] or Chapter 14 of [28].

To motivate our discussion of ends, we give two examples of manifolds. Each is a subset of the plane and inherits its topology and measure (area) from the plane.

$$\begin{aligned} X_1 &= (R \times [0,1]) \cup \left(\bigcup_{i=-\infty}^{+\infty} \left[i - \frac{1}{4}, i + \frac{1}{4} \right] \times [1, \infty) \right), \text{(Manhattan manifold)} \\ X_2 &= R \times [0,1] \, \text{(strip manifold)}. \end{aligned}$$

For $i = 1, 2$, let μ_i denote the restriction of planar Lebesgue measure (area) to X_i and let $t_i : X_i \to X_i$ be the restriction of unit horizontal translation

$(x, y) \longmapsto (x+1, y)$ to X_i. Note that X_2 is a simple horizontal strip while X_1 additionally has infinitely many vertical strips going up at each integer (hence its name). For each i, t_i belongs to $\mathcal{M}[X_i, \mu_i]$.

Definition 26 *An end e of a manifold X is a function which assigns to each compact subset K of X a non-empty unbounded component $e(K)$ of $X - K$ in such a way that*

$$K_1 \subset K_2 \text{ implies } e(K_2) \subset e(K_1).$$

The set of all ends of X is denoted $E[X]$.

The strip manifold X_2 has two ends, which we denote by $e_{-\infty}$ and e_∞, with $e_{-\infty}(I^2) = (-\infty, 0) \times [0, 1]$ and $e_\infty(I^2) = (1, \infty) \times [0, 1]$, where $I^2 = [0, 1]^2$. The Manhattan manifold X_1 has in addition ends e_i for each integer i corresponding to each of the vertical strips $\left[i - \frac{1}{4}, i + \frac{1}{4}\right] \times [1, \infty)$. For example we have $e_i([i-1, i+1] \times [0, 1]) = \left[i - \frac{1}{4}, i + \frac{1}{4}\right] \times [1, \infty)$.

The manifold X can be compactified by adjoining the set $E[X]$ of ends and defining, for each compact set $K \subset X$, a basic neighborhood of an end e_0 as $N_K(e_o) = e_o(K) \cup \{e \in E[X] : e(K) = e_0(K)\}$. With the relative topology on $E[X]$, the ends form a totally disconnected set. We now show how the measure μ on X induces a measure μ^* on $E[X]$.

Definition 27 *An OU measure μ on X induces a two valued (0 and ∞) measure μ^* on the algebra $\mathcal{Q}$ of clopen (closed and open) subsets Q of $E[X]$ as follows. If Q is a clopen set of ends, then set $\mu^*(Q) = 0$ if for some (sufficiently large) compact set K we have $\mu(Q(K)) < \infty$; otherwise set $\mu^*(Q) = \infty$. A single end e is said to have finite measure if for some compact set K we have $\mu(e(K)) < \infty$; otherwise we say it has infinite measure. The set of all ends of infinite measure, denoted $E^\infty[X]$, is a closed subset of $E[X]$.*

In the two examples (X_i, μ_i) given above, all the ends have infinite measure. However for area measure on the half strip $(0, \infty) \times [0, 1]$ there is still an infinite end (at infinity) but now also an end of finite measure. In addition to measures on the ends, there is also an induced homeomorphism.

Definition 28 *For any homeomorphism $h \in \mathcal{M}[X, \mu]$, define the measure preserving system $(E[X], \mathcal{Q}, \mu^*, h^*)$, by*

$$(h^*(e))(K) = h\left(e\left(h^{-1}(K)\right)\right),$$

for all $e \in E[X]$ and all compact sets K. We call the homeomorphism $h^* : E[X] \to E[X]$ the ***induced homeomorphism*** *of h. If h^* is the identity we say that h is* ***end-preserving****. If $\sigma : E[X] \to E[X]$ is such an induced homeomorphism, we define the compact-open topology closed subspace $\mathcal{M}_\sigma[X, \mu]$ of $\mathcal{M}[X, \mu]$ by*

$$\mathcal{M}_\sigma[X, \mu] = \{f \in \mathcal{M}[X, \mu] : f^* = \sigma\}.$$

To illustrate the notion of induced homeomorphism, we look again at the above examples $t_i \in \mathcal{M}[X_i, \mu_i]$. The translation t_1 of the Manhattan manifold induces the homeomorphism t_1^* of $\{e_{-\infty}, \ldots, e_{-2}, e_{-1}, e_0, e_1, \ldots, e_\infty\}$ given by $t_1^*(e_i) = e_{i+1}$, $t_1^*(e_{-\infty}) = e_{-\infty}$, and $t_1^*(e_\infty) = e_\infty$. The translation t_2 on the strip manifold is end preserving – t_2^* fixes both ends.

It will turn out that the typical dynamical properties in a closed subspace $\mathcal{M}_\sigma[X, \mu]$ will depend on the common dynamical system $(E[X], \mathcal{Q}, \mu^*, \sigma)$ which all such homeomorphisms induce on the ends. With this in mind we make the following definitions.

Definition 29 *Let $\sigma : E[X] \to E[X]$ be the induced homeomorphism h^* of some $h \in \mathcal{M}[X, \mu]$. The system $(E[X], \mathcal{Q}, \mu^*, \sigma)$ is called* ***compressible*** *if there is a clopen set $Q \in \mathcal{Q}$ such that $\mu^*(Q - \sigma Q) = 0$ and $\mu^*(\sigma Q - Q) = \infty$. Otherwise it is called* ***incompressible****. The system is called* ***ergodic*** *if for every clopen set $I \in \mathcal{Q}$ with $\mu^*(I \triangle \sigma I) = 0$, either $\mu^*(I) = 0$ or $\mu^*(E[X] - I) = 0$.*

As examples, note that the induced translation t_1^* of the ends of the Manhattan manifold X_1 is compressible, as t_1^* maps the clopen end set $\{e_0, e_1, \ldots, e_\infty\}$ into its proper subset $\{e_1, e_2, \ldots, e_\infty\}$, while t_2^* is incompressible. The induced translation t_1^* is ergodic, as none of its three invariant sets are clopen (the singletons $e_{-\infty}$ and e_∞ are not open and the set of e_i's is not closed). The induced translation t_2^* is not ergodic as the two ends are each clopen invariant sets if infinite measure.

3.3 Charge on the ends and recurrence

In this section we show that when a measure preserving homeomorphism generates a net flow of measure between two ends of infinite measure, it cannot be recurrent or ergodic. To do this we will define the *charge* that it induces on the ends. In subsequent sections we will show that ergodicity is generic when the induced charge is identically zero. The material in this section consists mainly of definitions with some properties needed to be checked. Details can be found [19] or Chapters 13 and 14 of [28].

We are mainly concerned with the net flow into an *end*, but we begin by defining the net flow into a *set* under an automorphism of the manifold. The following is essentially a definition of the *net* function, as its properties are easily checked.

Lemma 30 *Given an automorphism $g \in \mathcal{G}[X, \mu]$ and a measurable set V, define the net flow into V under g, denoted $net(g, V)$, by*

$$net(g, V) = \mu(V - g(V)) - \mu(g(V) - V),$$

setting $\infty - \infty = 0$ whenever this case arises. Then

1. *$net(g, V) = 0$ when $\mu(V) < \infty$.*
2. *g is μ-recurrent if and only if $net(g, V) = 0$ for all V.*

3. *net* (g, V) *is continuous in* g *in the weak topology.*

4. *If* $g \in \mathcal{M}[X, \mu]$ *and* g^* *is incompressible, then both terms in the definition of net* (g, V) *are finite for any* V.

It may be observed that conclusions 1 and 2 constitute a statement of the Poincaré Recurrence Theorem (that automorphisms preserving a finite measure are recurrent).

We can use this lemma to show that the translation t_1 of the Manhattan manifold (X_1, μ_1) is not ergodic and neither is any other homeomorphism with the same end action. This will be the first of our negative results showing that neither the Oxtoby-Ulam Theorem (Theorem 3) nor its generalization Theorem 23 can be extended without modification to noncompact manifolds. Recall that we showed t_1 induces a compressible homeomorphism of the ends $E[X_1]$. The consequences of this for t_1 are contained in the following result [19][28, Lemma 14.15].

Theorem 31 *If* $(E[X], \mathcal{Q}, \mu^*, \sigma)$ *is a compressible induced system, then no homeomorphism in* $\mathcal{M}_\sigma[X, \mu]$ *is* μ*-recurrent or ergodic.*

Proof. By the definition of compressibility, there is a clopen set Q of ends of X such that $\mu^*(\sigma Q - Q) = 0$ and $\mu^*(Q - \sigma(E[X] - Q)) = \infty$. It can be shown [28, Lemma 14.12] that for any homeomorphism $h \in \mathcal{M}_\sigma[X, \mu]$ we have $\mu(h(Q(K)) - Q(K)) < \infty$ and $\mu(Q(K) - h(Q(K))) = \infty$ for all sufficiently large compact sets K. Consequently $net(h, Q(K)) \neq 0$ and the previous lemma h cannot be μ-recurrent and hence cannot be ergodic. ∎

We also wish to show that the translation t_2 on the strip manifold (X_2, μ_2) is not ergodic and furthermore that it has a neighborhood in the compact-open topology which contains no ergodic homeomorphisms. To do this we will need to define the notion of the charge induced on the ends of a manifold by a homeomorphism. Roughly speaking, the charge on an end is the net flow of measure into that end.

Lemma 32 *Every homeomorphism* $h \in \mathcal{M}[X, \mu]$ *induces a charge (finitely additive signed measure)* $c = c_h$ *on the subalgebra* $\mathcal{I}$ *of* $\mathcal{Q}$*, consisting of all* h^**-invariant clopen sets of ends, by the formula*

$$c_h(I) = net(h, I(K)),$$

where $I \in \mathcal{I}$ *and* K *is a compact subset of* X*. The right hand side is constant for all sufficiently large* K*. The charge* c *has the following properties:*

(i) $c(E) = 0$

(ii) *If* $\mu^*(I) = 0$ *then* $c(I) = 0$

(iii) *The function* $c_h(I)$ *is continuous in* h*, and consequently the subspace* $\mathcal{M}_\sigma^c[X, \mu]$ *of* $\mathcal{M}_\sigma[X, \mu]$ *consisting of* h *with* $c_h = c$ *is closed. (Both statement are with respect to the compact-open topology.)*

To illustrate the notion of charge, we compute it for the translation t_1 on the strip manifold $X_1 = R \times [0,1]$. In the defining formula, take $I = \{e_\infty\}$, and $K = [2,3] \times [0,1]$. Consequently $I(K) = (3,\infty) \times [0,1]$, $t_1(I(K)) = (4,\infty) \times [0,1]$, and

$$c_{t_1}(e_\infty) = \mu((3,4] \times [0,1]) - \mu(\emptyset) = 1, c_{t_1}(e_{-\infty}) = -1.$$

The fact that the charge induced by the translation of the strip manifold is not identically zero means that the translation cannot be recurrent, nor can any other homeomorphism which is sufficiently close. This is made precise in the following simple result [28, Theorem 14.23].

Theorem 33 *If a homeomorphism $h \in \mathcal{M}_\sigma[X,\mu]$ induces a non-zero charge c_h, then h is not the limit, in the compact-open topology, of ergodic (or even μ-recurrent) homeomorphisms in $\mathcal{M}[X,\mu]$.*

Proof. Chose a σ-invariant clopen set I for which $c_h(I) \neq 0$. Consequently by part (iii) of the above result we also have $c_f(I) \neq 0$ for all f in some compact-open neighborhood of h. By the definition of charge, this means that $net(f, I(K)) \neq 0$ for all sufficiently large compact sets K. Consequently part 2 of Lemma 30 implies that f cannot be μ-recurrent (and hence it cannot be ergodic). ■

To summarize the easy results of this section, we have the following negative result (combining Theorems 31 and 33) on the impossibility of extending the Oxtoby-Ulam theory to arbitrary homeomorphisms of a sigma-compact manifold.

Corollary 34 *Let μ be an OU measure on a sigma-compact n-manifold X, $n \geq 2$. If $h \in \mathcal{M}[X,\mu]$ either induces a compressible homeomorphism h^* of the ends of the manifold, or induces a non-zero charge c_h on these ends, then it is not the limit of ergodic homeomorphisms in $\mathcal{M}[X,\mu]$, in the compact-open topology.*

This result sets up the deeper converse result, to be stated in the next section (as the main part 'if' part of Theorem 35), which says that if h has neither of these two problems then it is indeed the limit of ergodic homeomorphisms. Before proceeding to the positive results of the next section it is worth observing that if (X,μ) is Euclidean space R^n with volume measure, then every homeomorphism in $\mathcal{M}[X,\mu]$ is incompressible and induces a zero charge on the unique end of that manifold.

3.4 Conditions for generic ergodicity on noncompact manifolds

In this section we describe the results of two papers of the authors, [17] and [19], which answer the following questions regarding generic ergodicity on a sigma-compact measured manifold (X,μ). The questions ask to what extent the result of Oxtoby and Ulam for compact manifolds can be extended to sigma-compact manifolds. The questions are given order of decreasing strength of conclusion, that is, generic ergodicity in the whole space, generic ergodicity in a subspace, existence of ergodicity in a subspace, approximate ergodicity of a particular homeomorphism.

Question 1 For which measured manifolds (X, μ) does the Oxtoby-Ulam Theorem (Theorem 3) hold, in the sense that ergodicity is generic in the whole space $\mathcal{M}[X, \mu]$?

Question 2 For which induced end homeomorphisms $\sigma : E[X] \to E[X]$ is ergodicity generic in the subspace $\mathcal{M}_\sigma [X, \mu]$ of $\mathcal{M}[X, \mu]$?

Question 3 For which induced end homeomorphisms $\sigma : E[X] \to E[X]$ does there *exist* an ergodic homeomorphism in $\mathcal{M}_\sigma [X, \mu]$?

Question 4 If μ is an OU measure on a sigma-compact manifold X, under what conditions does there exist an ergodic homeomorphism in $\mathcal{M}[X, \mu]$?

Question 5 Which homeomorphisms in $\mathcal{M}[X, \mu]$ can expressed as the limit, in the compact-open topology, of ergodic homeomorphisms in $\mathcal{M}[X, \mu]$?

We will begin with question 5, since its answer (Theorem 35 below) can be used to answer the other questions. It is the converse of Theorem 34.

Theorem 35 *Let μ be an OU measure on a sigma-compact manifold X. Then a μ-preserving homeomorphism of X belongs to the closure, (with respect to the compact open topology) of the ergodic μ-preserving homeomorphisms, if and only if it induces an incompressible end homeomorphism and it induces an identically zero charge on the ends.*

If $\mathcal{E} \subset \mathcal{G}[X, \mu]$ denotes the set of ergodic automorphisms of an infinite Lebesgue space, it follows from Theorem 8 that $\mathcal{E} \cap \mathcal{M}[X, \mu]$ is a G_δ subset of $\mathcal{M}[X, \mu]$ with respect to the compact open topology. Consequently the above result is equivalent to saying that [28, Theorem 15.1]

Theorem 36 *The ergodic homeomorphisms $\mathcal{E} \cap \mathcal{M}[X, \mu]$ are generic in the closed subspace $\mathcal{M}^c_\sigma [X, \mu]$ of $\mathcal{M}[X, \mu]$ if and only if σ is incompressible and c is identically zero.*

If a measured manifold (X, μ) has only one end of infinite measure, then all the induced homeomorphisms from $\mathcal{M}[X, \mu]$ will fix this end and will be incompressible. Part (i) of Lemma 32 shows that the induced charges will all be identically zero. So for such manifolds, every homeomorphism in $\mathcal{M}[X, \mu]$ satisfies the two conditions in the Theorem above. On the other hand if (X, μ) has two or more ends of infinite measure there is always a homeomorphism, like the translation on the strip manifold, which is end preserving and flows a positive amount of mass from one of these ends to another one. Consequently it fails the second condition and therefore has a neighborhood which contains no ergodic homeomorphisms. These simple observations (with some checking of details) lead to the following answer to Question 1. (If (X, μ) has *no* ends of infinite measure then $\mu(X) < \infty$ and a stronger result obtains – see Theorem 43(i).)

Corollary 37 *A necessary and sufficient condition for ergodicity to be generic in the full space $\mathcal{M}[X, \mu]$ is that the underlying manifold (X, μ) has at most one end of infinite measure.*

Since Euclidean space $R^n, n \geq 2$, has one end ('at infinity'), Prasad's theorem [59] on the genericity of ergodicity in R^n can be seen as a special case of this result.

We now consider Question 2, for which the complete answer can be stated simply as follows.

Corollary 38 *Suppose $(E[X], \mathcal{Q}, \mu^*, \sigma)$ is the dynamical system induced on the ends by some homeomorphism $h \in \mathcal{M}[X, \mu]$. A necessary and sufficient condition that ergodicity is generic in $\mathcal{M}_\sigma[X, \mu]$ is that $(E[X], \mathcal{Q}, \mu^*, \sigma)$ is incompressible and ergodic.*

Proof. Suppose that σ is ergodic and incompressible. Let I be any σ-invariant clopen set of ends and let $f \in \mathcal{M}_\sigma[X, \mu]$. From the assumption of ergodicity it follows that either $\mu^*(I) = 0$ or $\mu^*(E - I) = 0$. In either case it follows from the additivity of charge that $c_f(I) + c_f(E - I) = c_f(E)$ which equals zero by part (i) of Lemma 32. Consequently $c_f(I)$ and $c_f(E - I)$ must both be zero. Since I was an arbitrary σ-invariant clopen set of ends it follows that c_f is identically zero. Consequently $\mathcal{M}_\sigma[X, \mu] = \mathcal{M}_\sigma^0[X, \mu]$, and Theorem 36 says that ergodicity is generic in the latter set.

If σ is compressible, the result follows from Theorem 31. If σ is incompressible but not ergodic, then there exist complementary σ-invariant sets I and J on which μ^* is infinite. Let $f \in \mathcal{M}_\sigma[X, \mu]$ be a homeomorphism which flows a positive mass from an infinite end in I to and infinite end in J. Then c_f will not be identically zero and consequently f will have a neighborhood containing no ergodic homeomorphisms. ∎

It follows from the above result that ergodicity is generic for measure preserving homeomorphisms of a manifold with finitely many ends of infinite measure which cyclically permute these ends – for example end reversing area preserving homeomorphisms of the strip manifold $R \times [0, 1]$.

For those preferring a purely topological condition on the end homeomorphism σ, we mention a related result.

Theorem 39 *If σ is transitive on E then ergodicity is generic in $\mathcal{M}_\sigma[X, \mu]$.*

Finally, we get to Question 3. The answer follows immediately from Theorem 35, since if $\sigma = h^*$ is incompressible, it can be shown that the subset $\mathcal{M}_\sigma^0[X, \mu]$ of $\mathcal{M}_\sigma[X, \mu]$ is nonempty.

Corollary 40 *The space $\mathcal{M}_\sigma[X,\mu]$ contains an ergodic homeomorphism if and only if σ is incompressible.*

Since the induced homeomorphism of the identity homeomorphism in $\mathcal{M}[X,\mu]$ is the identity homeomorphism id of the ends, which is incompressible, it follows that $\mathcal{M}_{id}[X,\mu]$ always contains an ergodic homeomorphism. Thus we have the following. We repeat our ongoing assumptions on X and μ only for emphasis.

Corollary 41 *If μ is an OU measure on a sigma-compact manifold X, then there always exists an ergodic μ-preserving homeomorphism of X.*

Thus while the genericity claim of the Oxtoby-Ulam Theorem cannot be extended to arbitrary sigma-compact manifolds (unless they have a single end), the existence claim of their theorem does indeed hold for such manifolds. Thus in their language we can say that the assumption of ergodicity for homeomorphisms of a sigma-compact manifold 'involves no topological contradiction'.

In this section we have shown how the answers to all the other questions can be obtained as corollaries to the answer (Theorem 35 – or the alternative formulation Theorem 36) to Question 5. Note that we have not given the proof of that result, which is difficult. We can however say a few words about the proof. Suppose we are given an arbitrary homeomorphism h which induces an incompressible end homeomorphism and zero charge on the ends, and a compact set K. Then there is an associated finite partition

$$X = K \cup P_1(K) \cup P_2(K) \cup \cdots \cup P_m(K), \tag{1}$$

where the P_i form a clopen partition of $E[X]$ and the sets $P_i(K)$ are the connected components of $X - K$. The main construction of the theorem produces an ergodic automorphism $g \in \mathcal{G}[X,\mu]$ which agrees setwise with h on the above partition and which is uniformly close to h on K. (Clearly a prior perturbation of the original h may be necessary if any of these sets are originally h-invariant.) The automorphism g is made to be ergodic by constructing it as a skyscraper over an ergodic base which contains K. In order to ensure that g is pointwise uniformly closes to h on K and agrees setwise with h on the given partition we use the following strengthening of our earlier result (Theorem 21) that the conjugacy class of an antiperiodic automorphism is dense in $\mathcal{G}$.

Corollary 42 *Let ρ be a totally bounded, connected metric on a Lebesgue probability space (X,Σ,μ) such that any nonempty open set has positive measure. Let $h,g \in \mathcal{G}[X,\mu]$, with g antiperiodic. Furthermore suppose Λ is a finite subalgebra of Σ such that $h_{/\Lambda}$ has no nontrivial periodic set and let C denote the union of all atoms of Λ whose image under h is connected and relatively compact. Then given any $\varepsilon > 0$ there is an $f \in \mathcal{G}[X,\mu]$ such that the conjugate $\bar{g} = f^{-1}gf$ satisfies $\rho(\bar{g}(x), h(x)) < \varepsilon$ for μ-a.e. $x \in C$ and $\bar{g}(A) = h(A)$ for all sets $A \in \Lambda$.*

Then we use a noncompact version of the Lusin Theorem for Measure Preserving Homeomorphisms (Theorem 9) which can be found in [28, Theorem

14.9] and which enables us to approximate the difference $\bar{g}^{-1}h$ by a μ-preserving homeomorphism which is small on K and leaves invariant all the elements of the algebra Λ generated by the partition (1). The details can be found in [19] or in Chapter 16 of [28].

3.5 Other properties in $\mathcal{M}[X, \mu]$

We have already shown in section 3.3 that the Oxtoby-Ulam Theorem on *genericity* of ergodicity in $\mathcal{M}[X, \mu]$ for compact X does not extend to arbitrary sigma-compact manifolds, although the *existence* of ergodicity in $\mathcal{M}[X, \mu]$ can be thus extended (Corollary 41). We now ask the same question for the more general result of the first author, Theorem 23. Recall that this asserted, for compact X, that if *any* measure theoretic property $\mathcal{F}$ is generic in $\mathcal{G}[X, \mu]$ then it is also generic in $\mathcal{M}[X, \mu]$. Of course the genericity result cannot be extended to sigma-compact manifolds either (for example $\mathcal{F}$ might be the property of ergodicity). In this section we show that any such property $\mathcal{F}$ *is* however generic in the subspace $\mathcal{M}_\sigma[X, \mu]$ of $\mathcal{M}[X, \mu]$ if the end homeomorphism σ has certain properties. Sufficient properties for this result to hold are that σ is the identity or that σ is *componentwise weak mixing* (a property we define later on). Since property $\mathcal{F}$ is represented by homeomorphisms in the nonempty subspace $\mathcal{M}_{id}[X, \mu]$, the existence question for the full space $\mathcal{M}[X, \mu]$ is answered positively: For any sigma-compact manifold (X, μ), the space $\mathcal{M}[X, \mu]$ contains a homeomorphism with any property which is generic (weak topology) for automorphisms of the underlying space. In the case that $\mu(X)$ is finite, we can say furthermore that homeomorphisms with property $\mathcal{F}$ are *generic* in $\mathcal{M}[X, \mu]$.

We now give the definition which we thought too technical for the above introduction: A self homeomorphism σ of a compact topological space is called *componentwise weak mixing* if for all clopen sets U, V, the set $\{k : \sigma^{-k}(U) \cap V \neq \emptyset\}$ contains arbitrarily long strings of consecutive integers.

We now state as a single result the results of three quite distinct theorems of the authors which have the common feature that they conclude that a property $\mathcal{F}$ is generic. Part (i) was proved in [17] (for manifolds with finitely many ends) and more generally in [19]. Part (ii) was proved in a joint paper with J. Choksi [15]. Part (iii) was proved in [18].

Theorem 43 *Let μ be an OU measure on a sigma-compact manifold X. Let $\mathcal{F}$ be any conjugate-invariant dense G_δ subset of $\mathcal{G}[X, \mu]$, with respect to the weak topology. Assume either*

(i) $\mu(X) < \infty$, *or*

(ii) $\sigma : E[X] \to E[X]$ *is the identity, or*

(iii) $\sigma : E[X] \to E[X]$ *is componentwise weak mixing.*

Then $\mathcal{F} \cap \mathcal{M}[X, \mu]$ is a dense G_δ subset of $\mathcal{M}^0_\sigma[X, \mu]$ with respect to the compact-open topology. Furthermore in case (i) $\mathcal{F} \cap \mathcal{M}[X, \mu]$ is a dense G_δ

subset of the full space $\mathcal{M}[X,\mu]$ *and in case (iii) it is a dense* G_δ *subset of* $\mathcal{M}_\sigma[X,\mu]$.

Part (i) is taken from [19], part (ii) is from [15], and part (iii) is from [18]. These three results have completely different proofs, none of which we will give in full here.

The significance of part (i) is that it shows that the obstruction in extending Theorem 23 to the noncompact measured manifolds (X,μ) is not so much the noncompactness of X but the nonfiniteness of μ. (A Borel measure on a compact space is necessarily finite.) An important portion of the proof of part (i) is the use of Corollary 42, as applied to the partition (1).

Part (ii) is proved using Theorem 24 and an end-preserving version of Theorem 9, in a manner somewhat analogous to the proof of the similar result (Theorem 25) for R^n (where any end-homeomorphism is the identity for the simple reason that there is a single end). In fact the hypothesis can be weakened to say that σ fixes an isolated end of infinite measure.

Part (iii) is the most technical and we shall not attempt to describe the proof here. However we will construct an example where the hypothesis applies, since the construction is informative about ends. First define X as $D-C$, where D is the unit disk $\{(x,y) : x^2+y^2 \leq 1\}$ in the plane and C is the standard Cantor ternary set lying on the line $[-1/2,+1/2]\times\{0\}$ along the x-axis. The ends E of X may be identified with the points in C, and the topology on the ends is that of the relative topology on C as a subset of the plane. Let μ be any OU measure on X which has no ends of finite measure. Let σ be any homeomorphism of the set $C=E$ which is topologically weak mixing, for example the two-shift when C is regarded as the countable product of a two-point set. The homeomorphism σ of C can be extended to a homeomorphism h of D using an extension theorem due to Antoine [29] and made μ-preserving using noncompact version of the Homeomorphic Measures Theorem (Theorem 1) due to Berlanga and Epstein. This homeomorphism $h\in\mathcal{M}[X,\mu]$ induces an end homeomorphism $h^*=\sigma$ which is componentwise weak mixing.

Since the identity homeomorphism on any manifold induces the identity homeomorphism of the ends and the identically zero charge on the ends, it follows that the space $\mathcal{M}^0_{id}[X,\mu]$ is never empty. Consequently part (ii) of Theorem 43 establishes an important existence result [15].

Theorem 44 *Let* μ *be an OU measure on a sigma-compact manifold* X. *Then for any property which constitutes a conjugate-invariant dense* G_δ *subset* $\mathcal{F}$ *of* $\mathcal{G}[X,\mu]$, *with respect to the weak topology, there is a* $\mu-$*preserving homeomorphism of* X *possessing this property.*

Theorem 43 establishes the existence of weak mixing homeomorphisms in the space $\mathcal{M}_\sigma[X,\mu]$ in the cases that $\mu(X)<\infty$, σ is the identity, or σ is componentwise weak mixing. What can we say in excluded cases such as the space of end-reversing area (λ) preserving homeomorphisms of the strip manifold X_2? To this end define $r : X_2 \to X_2$ to be the reversing or flip map $r(x,y)=$

$(-x, y)$, and note that its induced end homeomorphism r^* is the transposition of the two ends $e_{-\infty}$ and e_∞. We now prove the new result that $\mathcal{M}_{r^*}[X_2, \lambda]$ contains a topologically weak mixing homeomorphism h. (This means that for any open sets U_1, U_2, V_1, V_2, there is some m for which $h^m(U_i) \cap V_i$ are both nonempty.)

We will establish moreover that h has a stronger property which we call the *prime cycle condition*: Given any three open sets U, V, W, there are h-periodic points p and q in U, with relatively prime periods, whose orbits respectively enter V and W. It is easy to check that any homeomorphism satisfying this condition possesses the property we call *maximal chaos*, which entails the following three conditions:

1. The points periodic under h are dense.
2. h is transitive.
3. For any nonempty open set U, $\overline{lim}_i \text{diameter}\left(h^i(U)\right) = \text{diameter}(X)$.

We call this condition maximal chaos because condition 3 implies sensitive dependence on initial conditions with the maximum possible constant D of $\text{diameter}(X)/2$. (This is the maximum because h might have a fixed point at a center of X.) The definition of *sensitive dependence on initial conditions* (which, together with 1 and 2 defines Devaney-chaos) is the following weakening of condition 3:

3′ For some $D > 0$, for any $\varepsilon > 0$ and any $x \in X$, there is a $y \in X$ with $d(x, y) < \varepsilon$ and some $i \geq 1$ with $d\left(h^i x, h^i y\right) \geq D$.

We now give a new result on end-reversing maps of the strip manifold. The significance of this is due to the fact that r^* is not the identity (nor does it fix any end) and it is not componentwise weak mixing. As such, the dynamical properties of the subspace $\mathcal{M}_{r^*}[X_2, \lambda]$ are not covered by our more general theorems.

Theorem 45 *There exists a homeomorphism $h \in \mathcal{M}_{r^*}[X_2, \lambda]$ which satisfies the prime cycle condition. In particular, h has maximal chaos and is topologically weak mixing.*

Proof. Let K_i, $i = 1, 2, \ldots$ be the increasing sequence of rectangles $[a_i, b_i] \times [0, 1]$ whose union is the whole strip manifold $X_2 = R \times [0, 1]$. Choose the $a_i \to -\infty$ and the $b_i \to +\infty$ so that their difference $c_i = b_i - a_i$ is an integer power of 2 for even i and an integer power of 3 for odd i. Let $\varepsilon_0 > 0$ be given. View the reversing homeomorphism r, described above, as an element of $\mathcal{M}[K_0, \lambda]$. Using Theorem 15 (or a much simpler result [28, Lemma 4.4]) in the special case $\alpha = 0$ and only considering the first part of the conclusion about m_1, we obtain a homeomorphism $f_0 \in \mathcal{M}[X_2, \lambda]$ with support in K_0 and with $||f_0||_{K_0} < \varepsilon_0$ such that $f_0 r$ cyclically permutes the centers of some arbitrarily fine dydadic decomposition of K_0. Let C_0 denote the set consisting of this cyclic orbit of $f_0 r$.

Observe that the cardinality of C_0 is a power of 2. Since K_1 is invariant under f_0r we apply the same process to it, with respect to an arbitrarily small $\varepsilon_1 > 0$ and to an arbitrarily fine triadic decomposition of K_1 rather than a dyadic one. Applying Theorem 15 (or a much simpler result [28, Lemma 4.4]) to f_0r gives us a homeomorphism $f_1 \in \mathcal{M}[X_2, \lambda]$ with support in K_1 and with $||f_1||_{K_1} < \varepsilon_1$ such that $f_1 f_0 r$ cyclically permutes the centers of some arbitrarily fine triadic decomposition of K_1. Call this cyclically permuted set C_1 and observe that its cardinality is a power of 3. In this construction we may ensure that f_1 is the identity on C_0. We continue in this fashion, alternating between choosing dyadic and triadic decompositions. If the sequence of ε_i is chosen sufficiently small in terms of the modulus of continuity of the homeomorphism $f_{i-1}f_{i-2}\ldots f_1 f_0 r$, then this sequence will converge to a homeomorphism $h \in \mathcal{M}[X_2, \lambda]$. We claim that the construction of h ensures that it has the prime cycle condition. To show this, let U, V, and W be three nonempty open subsets of the strip manifold X_2. Choose j sufficiently large so that $U \cap K_j$, $V \cap K_j$, and $W \cap K_j$ are all nonempty. Choose an even i greater than j and sufficiently large so that the open sets $U \cap K_j$ and $V \cap K_j$ each contains on of the dyadic cubes whose centers are permuted under $f_i f_{i-1} \ldots f_0 f r$ in a cycle C_i with period an integer power of 2. By construction the points of C_i are also cyclically permuted under h. Similarly choose an odd k greater than j and sufficiently large so that the open sets $U \cap K_j$ and $W \cap K_j$ each contains one of the triadic cubes whose centers are permuted under $f_k f_{k-1} \ldots f_0 f r$ and also under h in a cycle C_k with period an integer power of 3. Since the set C_i intersects U and V, while the set C_k intersects U and W, the prime cycle property has been established for the homeomorphism h. ■

In [27] (see also [28] for this formulation) the authors have shown that in fact every sigma-compact measured manifold supports a maximally chaotic homeomorphism.

Theorem 46 *Let μ be an OU measure on a sigma-compact n-manifold X, $n \geq 2$. Then μ-preserving homeomorphisms of X with maximal chaos are dense in the space $\mathcal{M}^0_{id}[X, \mu]$. In particular, every sigma-compact manifold supports a homeomorphism with maximal chaos. Furthermore when X is compact, or more generally has at most one end of infinite measure, such homeomorphisms are dense in the full space $\mathcal{M}[X, \mu]$.*

Cairns and Kolganova [35] have shown that every compact manifold of dimension greater than one admits a faithful Devaney-chaotic action of every countably generated free group.

3.6 Topological Dynamics on the Plane

In this section we consider volume preserving homeomorphisms of the n-torus which have mean rotation zero, and their lifts to volume (λ) preserving homeomorphisms of R^n. In the case of the plane ($n = 2$) we can combine our results with Brouwer's Plane Translation Theorem to obtain short proofs of some well known fixed point theorems. We now restrict our attention to the case where the underlying base manifold X is the n-dimensional torus T^n $(n \geq 2)$ or the annulus (or cylinder) $A = S^1 \times I$. We assume they are represented as the unit cube with appropriate boundary identifications. Let $\hat{X}$ be the universal covering space of X, that is R^n or $R \times [0,1]$, respectively.

Define the mean rotation vector v of a volume preserving homeomorphism h of X by the formula

$$v\,(h) = \int_{\Omega} \left(\hat{h}\,(x) - x\right)\, d\lambda(x), \tag{2}$$

where $\hat{h}$ is a lift of h to the covering space $\hat{X}$, and $\Omega = [0,1] \times [0,1]$ is a fundamental domain. Since $\hat{h}$ is defined only up to unit translations in each direction, v is defined modulo one in each coordinate. Observe that according to this definition the rotation $x \mapsto x + w$ (taken mod 1 in each coordinate) will have mean rotation w. When X is the annulus A, the mean rotation vector will always have the second coordinate zero. In a series of articles ([22], [25], [26]), the authors have established a number a properties which are typical for *lifts* of mean rotation zero homeomorphisms of X. See also [24]. (For the properties we are considering here, it does not follow that if a base homeomorphism has the property then so will its lift.)

Theorem 47 *Let X denote the n-torus, $n \geq 2$, or the annulus $S^1 \times [0,1]$, and let $\hat{X}$ denote its universal covering space R^n or $R \times [0,1]$. In the closed subspace of $\mathcal{M}\,[X, \lambda]$ consisting of all homeomorphisms with mean rotation zero, those with a Devaney-chaotic lift to $\hat{X}$ are dense, with respect to the uniform topology.*

This result has particular application in the case $n = 2$ because of Brouwer's theorem [33] for planar homeomorphisms. We give only the special case that we need. Note that a translation of the plane can only have a periodic point in the case that it is the zero translation (identity), it which case it has a fixed point. The following says that all orientation preserving homeomorphisms are like translations in this respect.

Theorem 48 (Plane Translation Theorem) *If an orientation preserving homeomorphism of the plane R^2 has a periodic point, then it has a fixed point.*

We can combine the previous two results to obtain a simple proof of the following result.

Theorem 49 (Franks) *An area preserving homeomorphism f of the $2-$torus or annulus which is homotopic to the identity and has mean rotation zero has a fixed point.*

Before giving our short proof of this result it is worth tracing its history. It was conjectured by Arnold, and later proved (along with more general results) by Conley and Zehnder [37], that any *diffeomorphism* of the torus which has mean rotation zero must have at least three fixed points. Franks [42] gave a direct proof that weakening the smoothness hypothesis from diffeomorphism to homeomorphism, would still keep a fixed point. The authors [23] produced a combinatorial proof of Franks' result for a single fixed point, and then Flucher [40] showed that with Franks' hypotheses there are at least two fixed points. The existence of a third fixed point under these assumptions has recently been established by LeCalvez [50].

Proof of Franks' Theorem for the Torus. First observe that if a homeomorphism of the torus has a chaotic lift, then by definition that lift has a periodic point, and since it is orientation preserving Brouwer's Plane Translation Theorem implies it has a fixed point. Consequently the original torus homeomorphism also has a fixed point. It now follows from Theorem 47 that for the space of area-preserving orientation-preserving homeomorphisms of the torus with mean rotation zero, a dense subset possesses fixed points. It follows from the compactness of the torus that in fact every homeomorphism in the space has a fixed point. ■

A similar sort of proof can be given, using our techniques [21], of the following famous result (at least for a single fixed point). See [58], [32], [41], and [51].

Theorem 50 (Poincaré's Last Geometric Theroem) *Let f be an orientation preserving area preserving homeomorphism of the annulus $A = S^1 \times [0,1]$, with a lift $\hat{f}$ to $R \times [0,1]$ satisfying $\hat{f}_1(x,0) < x, \quad \hat{f}_1(x,1) > x$. Then the homeomorphisms $\hat{f}$ and f have fixed points.*

We return now to a brief discussion of possible extensions of our above result, Theorem 47, on chaotic R^n homeomorphisms which are lifts of torus homeomorphisms. This differs from our result in the last section on maximally chaotic homeomorphisms on noncompact manifolds in that it produces homeomorphisms which are spatially periodic. One may ask what other properties can be produced for such spatially periodic R^n homeomorphisms (that is, lifts of torus homeomorphisms). The most we are able to get in this direction is the following result for the subset of $\mathcal{M}[R^n, \lambda]$ consisting of *stationary* homeomorphisms. (A homeomorphism of R^n is called stationary if it is the limit, in the uniform topology, of homeomorphisms which are spatially periodic and have mean translation zero.) See [13] for a more detailed statement of this result.

Theorem 51 *Ergodic volume preserving homeomorphisms form a dense G_δ subset of the space of stationary homeomorphisms of R^n, $n \geq 2$, with respect to the uniform topology.*

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