The Number of 2-SAT Functions

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Abstract

Our aim in this paper is to address the following question: of the 2^{2^n} Boolean functions on n variables, how many are expressible as 2-SAT formulae? In other words, we wish to count the number of different instances of 2-SAT, counting two instances as equivalent if they have the same set of satisfying assignments. Viewed geometrically, we are asking for the number of subsets of the n-dimensional discrete cube that are unions of $(n-2)$-dimensional subcubes.

There is a trivial upper bound of $2^{4\binom{n}{2}}$, the number of 2-SAT formulae. There is also an obvious lower bound of $2^{\binom{n}{2}}$, corresponding to the monotone 2-SAT formulae. Our main result is that, rather surprisingly, this lower bound gives the correct speed: the number of 2-SAT functions is $2^{(1+o(1))n^2/2}$.

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1 Introduction

In this paper we are interested in the number of Boolean functions on n variables that are expressible as k-SAT formulae, for fixed k. For most of the paper we shall be concerned with the first non-trivial case, namely $k = 2$, although at the end we shall briefly turn our attention to general values of k.

For general background and information about the satisfiability problem, see for example the papers in [8], although this paper is self-contained. For now, we just need the following definitions. Let $\{x_1, \dots, x_n\}$ be a collection of n Boolean *variables*. With each variable x is associated a *positive literal* x and a *negative literal* $\overline{x}$. The literal $\overline{x}$ is True/False if the variable x is False/True. A *k-clause* or just *clause* is a set of k literals associated with different variables, and a *k-SAT formula* is a set $\{C_1, \dots, C_t\}$ of k-clauses. A *satisfying assignment* for a k-SAT formula is an assignment of True/False to each variable such that each clause C_i contains at least one True literal. A k-SAT formula gives rise to a Boolean function, namely one taking the value 1 whenever the input forms a satisfying assignment. A *k-SAT function* is one that arises from some k-SAT formula.

Our main task in this paper is to estimate the number $G(n)$ of 2-SAT functions with n variables. This problem was independently raised by Martin [13].

The function $G(n)$ is certainly exponential in n^2. Indeed, the number of 2-SAT functions is trivially at most the number of 2-SAT formulae, which is $2^{4\binom{n}{2}}$, while it is easy to see that a lower bound is $2^{\binom{n}{2}}$, the number of *monotone* formulae (those involving only positive literals) – as we will spell out in detail later, these all correspond to distinct functions.

So what is the growth speed of $G(n)$? Is there a constant c such that $G(n)$ is $2^{(c+o(1))n^2}$? We shall prove that, in fact, the very crude lower bound above is the correct one. In other words, we shall prove the following result.

Theorem 1.1

$$G(n) = 2^{(1+o(1))n^2/2}.$$

Another way to view this problem is in terms of subsets of the discrete cube $\mathbf{2}^n = \{0,1\}^n$. How many subsets of the cube can be represented as unions of subcubes of codimension 2 – i.e., sets of the form $\{\mathbf{x} \in \mathbf{2}^n : x_i = a, x_j = b\}$, where a, b are 0 or 1?

This is equivalent to our problem, since, if we identify the n variables with the co-ordinates of the cube, the complement of the set of satisfying assignments for a 2-SAT formula is exactly a union of codimension-2 subcubes. Therefore Theorem 1.1 implies that the number of such sets is $2^{(1+o(1))n^2/2}$.

At first glance, it seems as though this cannot be a difficult result, but it does not seem to be easy. In fact, even the value is rather counterintuitive (see remarks in the next section). It seems strange that the trivial lower bound is actually roughly correct.

We remark that the problem for codimension 1 (i.e., 1-SAT formulae) is trivial: a union of codimension-1 cubes, if not the entire cube, has complement that is a subcube. Since the number of subcubes is 3^n (for each coordinate i, either fix $x_i = 1$, or fix $x_i = 0$, or allow coordinate i to vary), the number of 1-SAT functions on n variables is exactly $3^n + 1$. It is remarkable that codimension 2 is already non-trivial.

The plan of the paper is as follows. After some preliminaries in Section 2, we turn to the proof of Theorem 1.1 in Sections 3 and 4. In Section 5 we turn our attention to the problem for codimensions greater than 2. We prove a monotonicity result that implies a (weak) bound for codimension k. The problem for general codimension seems significantly harder than that for 2: we make some general remarks about this question in Section 6.

2 Preliminaries

In this Section we make a simple preliminary reduction, and then give two particular families of formulae that it may help the reader to bear in mind during subsequent work.

The function 0, corresponding to the empty set of satisfying assignments, is called the *trivial function*. As in the case of 1-SAT, it is important to distinguish this function, if only since the vast majority of 2-SAT formulae give rise to the trivial function.

The *spine* of a non-trivial 2-SAT function S is the set of literals that are True in all satisfying assignments for S. Note that, for any variable x, the spine of a non-trivial S contains at most one of x and $\overline{x}$; we can think of the *variable* x as being in the spine. For a 2-SAT function S with empty spine, we say that a pair (u, v) of literals is *associated* if $u \Leftrightarrow v$ is True in every satisfying assignment for S. Note that association is an equivalence relation, and also that (u, v) is associated if and only if $(\overline{u}, \overline{v})$ is; again we can think of the corresponding variables as being associated. We say that a 2-SAT function is *elementary* if it has empty spine and no associated pairs. Let $H(n)$ be the number of elementary 2-SAT functions on n variables.

Given any 2-SAT function S on n variables, we reduce it to an elementary 2-SAT function by ignoring all variables in the spine, and all but one variable (say the lowest numbered) in each association equivalence class; then we compress the labels on the remaining variables to get an elementary 2-SAT function on $x_1, \ldots, x_m$ for some $m \leq n$. An elementary 2-SAT function on $x_1, \ldots, x_{n-k}$ arises from at most $\binom{n}{k}(2n - 2k + 2)^k$ 2-SAT functions via this procedure – the original function is specified by choosing the labels of the variables to be removed and then, for each removed variable x, choosing either to put x or $\overline{x}$ in the spine, or to associate x with some remaining literal.

Therefore we have

$$H(n) \leq G(n) \leq 1 + \sum_{k=0}^{n} H(n-k)\binom{n}{k}(2n - 2k + 2)^k.$$

(Here the extra 1 is for the trivial function.)

We shall prove that $H(m) = 2^{(1+o(1))m^2/2}$, from which Theorem 1.1 will follow immediately. This corresponds to the intuitive idea that 'most' 2-SAT functions should be elementary.

Let us explain in detail why $H(n) \geq 2^{\binom{n}{2}}$, which is the claimed lower bound on $G(n)$, by returning to the family of *monotone* 2-SAT formulae, i.e., those in which only positive literals appear in the clauses.

There are $2^{\binom{n}{2}}$ monotone formulae, corresponding to the subsets of the set of clauses containing two positive literals. We claim that each subset gives a different function. Indeed, for any monotone 2-SAT formula, the assignment setting all but two variables True, and x and y False, is a satisfying assignment if and only if $\{x, y\}$ is not a clause. Thus different sets of clauses give different sets of satisfying assignments. Note that all the functions arising are elementary, since setting just one variable False always gives a satisfying assignment.

We remark that we may expand this set of functions slightly by allowing a "relabelling" of positive and negative literals. To be precise, let A be any set containing exactly one of each pair $(x, \overline{x})$, and let $\mathcal{C}_A$ be the set of formulae in which only literals from A appear. As above, the formulae in each $\mathcal{C}_A$ are all inequivalent. Also, provided every literal in A does appear, the set A can be recovered from the function, and therefore we have

$$G(n) \geq H(n) \geq 2^n \left(2^{\binom{n}{2}} - n2^{\binom{n-1}{2}}\right) = (1 - o(1))2^{n(n+1)/2}.$$

At this point, we introduce another family of functions; this family is not quite as large, but its prescription seems to be another "local maximum" among families of 2-SAT functions. To construct this family, we divide the variables into two classes U and V, as equally as possible. For each pair $u \in U$, $v \in V$, we take at most one of the clauses $\{u, v\}$ and $\{u, \overline{v}\}$. We take no other clauses. There are $3^{\lfloor n^2/4 \rfloor}$ formulae satisfying this prescription; we claim they all give rise to different functions. Indeed, consider setting every variable in U True except for the variable u, which is set False. All clauses except those containing u are satisfied; for those that do contain u, we can satisfy each in turn by setting the other literal in the clause True; if neither $\{u, v\}$ nor $\{u, \overline{v}\}$ is a clause then we can set v either True or False. We now see that we can recover the formula from the set of satisfying assignments.

We can again expand this family by allowing a choice of the set U, and also a relabelling of the positive and negative literals in U, but its size remains $3^{(1+o(1))n^2/4} \ll 2^{n^2/2}$.

Although this family is smaller than the one arising from the monotone formulae, it seems to us that any proof of Theorem 1.1 needs to in some way take account of it, and we urge the reader to follow our discussion of both of these families in later sections.

3 Reduction to a problem on coloured graphs

Our purpose in this section is to reduce the problem of bounding $H(n)$ to that of counting a certain family of coloured graphs. The first step is to establish a bijection between elementary 2-SAT functions and a certain class of partial orders. This step is fairly standard; for instance the bijection is essentially the same as that between 2-SAT functions and a class of digraphs given by Aspvall, Plass and Tarjan [1] (or see, for instance, Section 9.2 of Papadimitriou [14]).

As we shall see, extra structure arises since the 2-SAT functions are here assumed to be elementary.

Given a 2-SAT formula F giving rise to an elementary function S_F, we construct a partial order $P(F)$ on the set $\{x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$ by setting $u < v$ if the clause $\{\overline{u}, v\}$ appears in F (so that $u \implies v$ is True in any satisfying assignment for F), and taking the transitive closure of this relation. (We shall justify the assertion that $P(F)$ is a partial order shortly.) Thus we set $u < v$ if and only if there is a sequence of literals $u = u_1, u_2, \ldots, u_k = v$ such that each clause $\{\overline{u_i}, u_{i+1}\}$ appears in F. We see that, if $u < v$, then $u \implies v$ is True in all satisfying assignments for F.

We note the following properties of $P(F)$. Firstly, we have $u < v$ if and only if $\overline{v} < \overline{u}$, since a sequence of clauses certifying one relation also certifies the other. Next we see that each pair u and $\overline{u}$ of literals is incomparable in $P(F)$, since if $\overline{u} < u$ then u is True in all satisfying assignments, i.e., u is in the spine of S_F. Also we cannot have both $u < v$ and $v < u$, since then the pair (u, v) would be associated; therefore $P(F)$ is indeed a partial order.

It is clear that a satisfying assignment for F is exactly an up-set in $P(F)$ consisting of just one of each pair of literals. We also claim that, unless $u < v$ in $P(F)$, there is a satisfying assignment containing u and $\overline{v}$. Indeed, take the up-set U of all literals above either u or $\overline{v}$ in $P(F)$: if there is some literal x such that $u < x$ and $\overline{v} < \overline{x}$, then also $x < v$ and so $u < x < v$ in $P(F)$. We can now extend the up-set U one element at a time so that it contains exactly one of each pair of literals.

Therefore we have that $u \implies v$ is True in all satisfying assignments for F if and only if $u < v$ in $P(F)$. This tells us that the partial order $P(F)$ depends only on the function S_F – from now on we shall denote the partial order corresponding to the 2-SAT function S by $P(S)$ – and that elementary 2-SAT functions are in 1-1 correspondence with partial orders P on $\{x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$ such that

(a) $u < v$ if and only if $\overline{v} < \overline{u}$,

(b) each pair $u, \overline{u}$ is incomparable.

Hence $H(n)$ is just the number of partial orders satisfying (a) and (b).

We also see that there is a unique minimal formula for each elementary 2-SAT function S, namely that consisting of those clauses $\{\overline{u}, v\}$ where $u < v$ is a covering relation in $P(S)$ (i.e., there is no w with $u < w < v$). Likewise there is a unique maximal formula, consisting of all clauses $\{\overline{u}, v\}$ where $u < v$ in $P(S)$.

We have thus altered our problem to that of counting a certain class of partial orders. This seems already to be progress – notice that non-elementary 2-SAT functions do not give rise to partial orders – but the main elements of the proof are yet to come.

The problem is related to that of asymptotically enumerating the set of all partial orders, for which see Kleitman and Rothschild [12] and Brightwell, Promel and Steger [6]. Unfortunately, we have been unable to apply the methods of those papers here, or modify them to help, so we are forced to use a quite different approach.

Let us pause to consider our two example families from the previous section. If F is a monotone 2-SAT formula, then $P(F)$ is a bipartite partial order, with the negative literals as minimal elements and the positive literals as maximals. Each clause $\{u, v\}$ gives us the two relations $\overline{u} < v$ and $\overline{v} < u$ in $P(F)$.

For a formula F in our second family, we obtain a partial order $P(F)$ with three layers. The literals $\overline{u}$, for $u \in U$, are minimal elements, while the positive literals $u \in U$ are maximal. All the literals $v, \overline{v}$, with $v \in V$, are incomparable. Each clause $\{u, w\}$ gives us the two relations $\overline{u} < w$ and $\overline{w} < u$.

A *coloured graph* G is a graph on a vertex set V together with a 2-colouring (red and blue) of the edges of G. The next step is to encode most of the information of the partial order as a coloured graph. Our reduction between partial orders and coloured graphs is not 1–1; this loss of "information" is more than offset by the fact that the class of coloured graphs we deal with is simple to describe.

Given a partial order P on $\{x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$ satisfying (a) and (b) above, we construct a coloured graph $G(P)$ on vertex set $\{x_1, \ldots, x_n\}$ by putting a **red** edge between x and y if either $x < \overline{y}$ or $\overline{y} < x$ is a covering relation of P, and a **blue** edge between x and y if either $x < y$ or $y < x$ is a covering relation of P.

We claim that no edge xy is coloured both red and blue: if we have $x < \overline{y}$ and $x < y$, then also $\overline{y} < \overline{x}$ by (a), and so $x < \overline{x}$, violating (b) – other possibilities are ruled out similarly.

Observe that, given the coloured graph $G(P)$, we can recover the *covering graph* of the partial order P, i.e., the set of pairs $\{u, v\}$ such that either $u < v$ or $v < u$ is a covering relation of P. Indeed, a red edge between x and y corresponds to $\{x, \overline{y}\}$ and $\{\overline{x}, y\}$ being edges of the covering graph, while a blue edge corresponds to edges $\{x, y\}$ and $\{\overline{x}, \overline{y}\}$.

We next claim that there is no triangle in $G(P)$ with an odd number of blue edges. Indeed, if xy, yz and xz are all blue edges, then $\{x, y, z\}$ would form a triangle in the covering graph of P, which is not possible. Similarly if xy is a blue edge and yz and xz are red edges, then $\{x, y, \overline{z}\}$ is a triangle in the covering graph of P.

In some sense, our definition of the colouring is highly artificial, since it depends on the labelling of literals as positive and negative. However, the parity of the number of blue edges in any triangle is unchanged on relabelling any pair of literals, so the property of having no triangle with an odd number of blue edges is invariant under relabelling. This choice of a colouring, allowing us to consider this relatively simple property of graphs with just two colours on the edges, is crucial to the proof.

Another important observation is that, given the graph $G(P)$, and any linear extension of P, we can recover P. Indeed the graph $G(P)$ tells us the covering graph of P, and the linear extension tells us the orientation of each covering relation.

Therefore the number of partial orders P satisfying (a) and (b), which is $H(n)$, is at most $(2n)!$ times the number $J(n)$ of coloured graphs on n vertices in which every triangle has an even number of blue edges. So we aim to show that $J(n) = 2^{(1+o(1))n^2/2}$, which will imply Theorem 1.1.

Let us again consider our two example families, to see some large families of coloured graphs in which every triangle has an even number of blue edges.

The family of monotone 2-SAT formulae translates to the family of all graphs with only red edges. These all satisfy the required property rather trivially, and there are clearly $2^{\binom{n}{2}}$ such coloured graphs. Allowing the relabelling of some positive literals as negative extends this to the family of all graphs, and all partitions of the vertex set into two parts, with an edge coloured blue if and only if its ends are in different parts.

Our second family translates to the set of all bipartite graphs (with a fixed vertex partition into two classes as nearly equal in size as possible), with edges coloured arbitrarily. There are $3^{\lfloor n^2/4 \rfloor}$ such coloured graphs. Obviously we can actually extend this to the set of all triangle-free graphs coloured arbitrarily – but this does not increase the size of the family significantly, since almost every triangle-free graph is bipartite (this is due to Erdős, Kleitman and Rothschild [9]). Note however that not all coloured triangle-free graphs actually arise from 2-SAT functions (this follows from the fact that not all triangle-free graphs are covering graphs).

4 Counting coloured graphs

We now come to the heart of the proof. To motivate what comes next, let us observe that the property (of coloured graphs) that every triangle has an even number of blue edges is a hereditary property: if a coloured graph G has the property then so does every induced subgraph. This suggests that we might try to use a method invented by Bollobás and Thomason [5] to find the growth rate of the number of ordinary (uncoloured) graphs having a given hereditary property. Our general approach will be similar to theirs.

We need a version of Szemerédi's Uniformity Lemma (see, e.g., Bollobás [2]) for coloured graphs.

For a coloured graph G, and disjoint subsets Y, Z of $V(G)$, the *red-density* $d_r(Y,Z)$ is defined as the number of red edges between Y and Z divided by $|Y|\,|Z|$. The *blue-density* $d_b(Y,Z)$ is defined analogously.

Given a coloured graph G and $\varepsilon > 0$, an *ε-uniform pair* in G is a pair of subsets (Y,Z) of $V(G)$ such that, for any $S \subset Y$ and $T \subset Z$, with $|S| \geq \varepsilon|Y|$ and $|T| \geq \varepsilon|Z|$, we have

$$|d_r(S,T) - d_r(Y,Z)| < \varepsilon, \quad |d_b(S,T) - d_b(Y,Z)| < \varepsilon.$$

Now, given G and ε, an *ε-uniform partition* of G is a partition of $V(G)$ into sets $Y_0, Y_1, \ldots, Y_m$ with $|Y_0| \leq \varepsilon|V(G)|$, and $|Y_1| = \cdots = |Y_m|$, such that all but at most εm^2 of the pairs (Y_i, Y_j) with $1 \leq i < j \leq m$ are ε-uniform.

Theorem 4.1 *For any $\varepsilon > 0$, and any integer k, there is some $K = K(k,\varepsilon)$ such that every coloured graph G on at least k vertices has an ε-uniform partition into sets $Y_0, Y_1, \ldots, Y_m$ for some m with $k \leq m \leq K$.*

We omit the proof of this result – it may be found for example in Janson, Luczak and Ruciński [11], or alternatively it may be obtained by following almost word for word the standard proof of Szemerédi's Uniformity Lemma, e.g., as set out in Bollobás [2].

We shall also need the following lemma of Furedi [10].

For $G = (V, E)$ a graph, let G^2 be the "proper square" of G, i.e., the graph on V with xy an edge if and only if there is some $z \in V$ such that xz and zy are both edges of G. For G a graph, set $e(G)$ equal to the number of edges of G.

Lemma 4.2 *For any graph G on n vertices,*

$$e(G^2) \geq e(G) - \lfloor n/2 \rfloor.$$

For completeness, we give a proof.

Proof. Suppose the result is false, and let G be a minimal counterexample. Note first that G is connected, since otherwise some component will be a smaller counterexample.

Suppose that G is regular, of degree d. Then there are exactly $n\binom{d}{2}$ paths of length 2 (counting xzy and yzx as the same path), and each pair of adjacent vertices in G^2 is joined by at most d such paths, so the number of edges of G^2 is at least

$$\left\lceil \frac{n(d-1)}{2} \right\rceil = \frac{nd}{2} - \left\lfloor \frac{n}{2} \right\rfloor = e(G) - \left\lfloor \frac{n}{2} \right\rfloor,$$

so G is not after all a counterexample.

Therefore G is connected but not regular, so we can find adjacent vertices x and y with $d_G(y) > d_G(x) \geq 1$. Then $d_{G^2}(x) \geq d_G(y) - 1 \geq d_G(x)$. Since $(G - x)$ is not a counterexample, we have that

$$e(G^2) \geq e((G-x)^2) + d_{G^2}(x) \geq e(G-x) - \lfloor (n-1)/2 \rfloor + d_G(x) \geq e(G) - \lfloor n/2 \rfloor,$$

again contradicting the assertion that G is a counterexample. □

Recall that $J(n)$ is the number of coloured graphs on n vertices such that every triangle has an even number of blue edges. The following result will imply Theorem 1.1.

Theorem 4.3 *Given any $\delta > 0$, there is an n_0 such that, for $n \geq n_0$, $J(n) \leq 2^{n^2/2 + \delta n^2}$.*

Proof. Choose $\varepsilon < \frac{1}{2}$ suitably small and k suitably large in terms of δ (we shall be more specific later). Take $K = K(k, \varepsilon)$ as in the statement of Theorem 4.1, and choose $n_0 \geq k$ sufficiently large that $K < 2^{\delta n_0/4}$ and $5^{K^2/2} < 2^{\delta n_0^2/4}$. Let G be any coloured graph on $n \geq n_0$ vertices, and take an ε-uniform partition into sets $Y_0, Y_1, \ldots, Y_m$ $(k \leq m \leq K)$, as guaranteed by Theorem 4.1.

Suppose there are three parts Y_i, Y_j, Y_ℓ of the partition such that all the pairs (Y_i, Y_j), (Y_i, Y_ℓ) and (Y_j, Y_ℓ) are ε-uniform, and

$$d_r(Y_i, Y_j) \geq 2\varepsilon,\ d_b(Y_i, Y_j) \geq 2\varepsilon,\ d_b(Y_i, Y_\ell) \geq 2\varepsilon, \text{ and } \max(d_r(Y_j, Y_\ell), d_b(Y_j, Y_\ell)) \geq 2\varepsilon.$$

We claim that there is then a triangle xyz with $x \in Y_i$, $y \in Y_j$, $z \in Y_\ell$ and an odd number of blue edges. Indeed, if $d_b(Y_j, Y_\ell) > 2\varepsilon$ then we shall find such a triangle with all three edges blue, while if $d_r(Y_j, Y_\ell) > 2\varepsilon$ then xz will be the only blue edge. The two cases are obviously identical, so we concentrate on the first, where all three blue-densities are at least 2ε and we are looking for a blue triangle.

This is a very routine application of (some form of) the Uniformity Lemma. Let U be the set of vertices of Y_i sending fewer than $\varepsilon|Y_j|$ blue edges to Y_j, and W the set of vertices of Y_i sending fewer than $\varepsilon|Y_\ell|$ blue edges to Y_ℓ. Since the pairs are ε-uniform and $d_b(U, Y_j) < \varepsilon$, $d_b(W, Y_\ell) < \varepsilon$, we have $|U|, |W| < \varepsilon|Y_i|$, so we can find some vertex $x \in Y_i \setminus (U \cup W)$. Now let S be the set of vertices in Y_j sending a blue edge to x, and T be the set of vertices in Y_ℓ sending a blue edge to x. By choice of x, we have that $|S| \geq \varepsilon|Y_j|$ and $|T| \geq \varepsilon|Y_\ell|$, and therefore $d_b(S, T) > d_b(Y_j, Y_\ell) - \varepsilon \geq \varepsilon$. In particular there is some blue edge yz from S to T, and this gives the required blue triangle.

Suppose from now on that there is no triangle in G with an odd number of blue edges. We label each pair according to the densities.

- (Y_i, Y_j) is *rich* if it is ε-uniform and $d_b(Y_i, Y_j), d_r(Y_i, Y_j) \geq 2\varepsilon$.
- (Y_i, Y_j) is *red* if it is ε-uniform and $d_r(Y_i, Y_j) \geq 2\varepsilon > d_b(Y_i, Y_j)$.
- (Y_i, Y_j) is *blue* if it is ε-uniform and $d_b(Y_i, Y_j) \geq 2\varepsilon > d_r(Y_i, Y_j)$.
- (Y_i, Y_j) is *sparse* if it is ε-uniform and $d_b(Y_i, Y_j), d_r(Y_i, Y_j) < 2\varepsilon$.
- (Y_i, Y_j) is *bad* if it is not ε-uniform.

What we need from the previous argument is the weaker assertion that, if (Y_i, Y_j) and (Y_i, Y_ℓ) are both rich, then (Y_j, Y_ℓ) is either sparse or bad.

Let H be the graph on the set $\{Y_1, \ldots, Y_m\}$ defined by setting the rich pairs adjacent. Then every edge of H^2 is either a sparse or a bad pair. Applying Lemma 4.2 to H tells us that the number of rich pairs is at most the number of sparse pairs, plus the number of bad pairs, plus $m/2$.

Leaving the proof briefly, let us think about our two families of examples. In a coloured graph arising from a monotone formula, we can easily take an ε-uniform partition so that every pair is blue (or, if some literals are relabelled, every pair is either blue or red). In a coloured graph arising from a member of our second family, half the pairs are rich and half are sparse. The key idea is that knowing that most pairs are red or blue cuts down the number of possibilities for the graph. On the other hand, if there are many rich pairs, then there are practically as many sparse pairs, and this cuts down the number of possibilities even more.

Let us now return to the proof, and give an upper bound on the number $J(n)$ of coloured graphs G on $n \geq k$ vertices such that no triangle has an odd number of blue edges.

The number of partitions of the vertex set $\{1, \ldots, n\}$ into sets $Y_0, Y_1, \ldots, Y_m$ with $k \leq m \leq K$ and $|Y_1| = \cdots = |Y_m| = q$ is at most K^n, which by choice of n_0 is at most $2^{\delta n^2/4}$. For any such partition, we can label each pair (Y_i, Y_j) with $1 \leq i < j \leq m$ as rich, red, blue, sparse or bad in at most $5^{\binom{m}{2}} < 5^{K^2/2} < 2^{\delta n^2/4}$ ways. We can restrict attention to the cases where there are at most εm^2 bad pairs, some number s of sparse pairs, and at most $s + \varepsilon m^2 + m/2$ rich pairs.

Given a suitable partition, and a prescription for the type of each pair, how many graphs are there satisfying the various uniformity conditions? We know nothing about the edges inside the parts, or with one end in Y_0, or between bad pairs, and nothing of any use to us about the edges between rich pairs. This accounts for at most

$$mq^2/2 + \varepsilon n^2 + \varepsilon m^2 q^2 + (s + \varepsilon m^2 + m/2)q^2 \leq sq^2 + n^2/k + 3\varepsilon n^2$$

pairs of vertices, since $mq \leq n$ and $mq^2 \leq n^2/m \leq n^2/k$, so there are at most $3^{sq^2+n^2/k+3\varepsilon n^2}$ choices for these categories of edges.

The number of choices for edges across each sparse pair (Y_i, Y_j) is at most

$$\binom{q^2}{2\varepsilon q^2}^2 \leq \left(\frac{e}{2\varepsilon}\right)^{4\varepsilon q^2} = 2^{2\eta q^2},$$

where $\eta = \eta(\varepsilon) = 2\varepsilon \log_2(e/2\varepsilon)$, while the number of choices for the edges across each red or blue pair is similarly at most $2^{q^2(1+\eta)}$.

The total number of choices for the graph is therefore at most

$$\begin{aligned} &2^{\delta n^2/4}\, 2^{\delta n^2/4} 3^{sq^2+n^2/k+3\varepsilon n^2}\, 2^{2s\eta q^2}\, 2^{q^2(1+\eta)(m^2/2-2s)} \\ &\quad \leq 2^{q^2 m^2/2}\, 2^{\delta n^2/2}\, (3/4)^{sq^2}\, 3^{n^2(1/k+3\varepsilon+\eta/2)} \\ &\quad \leq 2^{n^2/2+\delta n^2}, \end{aligned}$$

provided ε and k were chosen so that $1/k + 3\varepsilon + \eta(\varepsilon)/2 \leq \delta/4$. This completes the proof. □

Let us now tidy up the proof of our main result.

Proof of Theorem 1.1. We have proved that the number $H(n)$ of elementary 2-SAT functions on n variables satisfies

$$H(n) \leq (2n)!\, J(n) = 2^{(1+o(1))n^2/2}.$$

Recall that

$$G(n) \leq 1 + \sum_{k=0}^{n} \binom{n}{k} (2n - 2k + 2)^k H(n - k).$$

We already know that $G(n) \geq 2^{(1+o(1))n^2/2}$. For the upper bound on $G(n)$, take any $\varepsilon > 0$, and choose m_0 such that $H(m) \leq 2^{(\frac{1}{2}+\varepsilon)m^2}$ for $m \geq m_0$. Note that, crudely, $H(m) \leq 2^{2m^2}$. Then we have, for $n \geq m_0$,

$$\begin{aligned} G(n) &\leq 1 + \textstyle\sum_{k=0}^{n-m_0} \binom{n}{k}(2n+2)^k 2^{(\frac{1}{2}+\varepsilon)(n-k)^2} + \sum_{k=n-m_0}^{n} \binom{n}{k}(2n)^k 2^{2(n-k)^2} \\ &\leq 1 + 2^{(\frac{1}{2}+\varepsilon)n^2} \textstyle\sum_{k=0}^{n-m_0} \binom{n}{k}(2n+2)^k 2^{-(\frac{1}{2}+\varepsilon)nk} + (4n)^n 2^{2m_0^2} \\ &\leq 1 + 2^{(\frac{1}{2}+\varepsilon)n^2} \left(1 + (2n+2)\cdot 2^{-n/2}\right)^n + (4n)^n 2^{2m_0^2}, \end{aligned}$$

which is at most $2^{(\frac{1}{2}+2\varepsilon)n^2}$ for n sufficiently large. □

5 Monotonicity

In this Section we turn our attention to the number of k-SAT functions, for k greater than 2. As is usual for k-SAT, the methods we have been using for 2-SAT break down completely in this case.

However, we are able to prove a weak bound for the number of k-SAT functions. The key idea is that of monotonicity, which we will illustrate for the case of 2-SAT. Our first aim will be to show that $\log G(n)/\binom{n}{2}$ is a decreasing function. Technically, we do not make use of this above, but it is interesting to know that a bound for some fixed n implies a bound for all greater n. The proof also goes through for k-SAT, and, rather remarkably, it gives an upper bound for that problem that is a considerable improvement on the trivial upper bound.

Viewed geometrically, it turns out that such monotonicity results are tailor-made for the uniform covers theorem of Shearer [7] (for several related results see Bollobás and Thomason [4]).

Theorem 5.1 *Let $M_1, \dots, M_r$ be subsets of $\{1, \dots, n\}$ such that every $x \in \{1, \dots, n\}$ belongs to exactly k of the M_i, and for each i write π_i for the projection from $\mathbf{2}^n$ to $\mathbf{2}^{|M_i|}$ obtained by ignoring those coordinates not in M_i. Then for any $A \subset \mathbf{2}^n$ we have*

$$|A|^k \leq \prod_{i=1}^{r} |\pi_i(A)|.$$

Let us now prove monotonicity for 2-SAT.

Theorem 5.2 *Let the constants $c_2, c_3, \dots$ be defined by*

$$G(n) = 2^{c_n \binom{n}{2}}.$$

Then $c_2 \geq c_3 \geq \cdots$.

Proof. Fix $1 < m < n$; we will show that $c_m \geq c_n$.

Given a 2-SAT function S on n variables, let F_S denote the unique maximal formula giving rise to S; i.e., F_S is the set of all clauses that are True in all satisfying assignments for S. This maximal formula F_S is characterized as the only formula for S satisfying the 'closure' condition: whenever some clauses in F_S imply another clause, then that clause is in F_S as well.

Now we view $\mathbf{2}^{4\binom{n}{2}}$ as a copy of the power-set of the set of all clauses, so that formally each F_S is an element of $\mathbf{2}^{4\binom{n}{2}}$.

Our aim is to estimate the size $G(n)$ of $\mathcal{G}_n = \{F_S : S \text{ is a 2-SAT function on } n \text{ variables}\}$. We will accomplish this by considering some projections of $\mathcal{G}_n$ onto lower-dimensional cubes. Indeed, given a fixed m-set M of variables, consider the projection π_M from $\mathbf{2}^{4\binom{n}{2}}$ to $\mathbf{2}^{4\binom{m}{2}}$ obtained by restricting to those coordinates that correspond to clauses involving only variables from M. Thus $\pi_M(F_S)$ consists of all clauses $\{u, v\}$ that involve only variables from M and are True in all satisfying assignments of S.

Now, a moment's thought shows that $\pi_M(F_S)$ satisfies the closure condition, so it is of the form F_T, for some 2-SAT function T on the m variables in M. Furthermore, each such F_T can arise, for instance as $\pi_M(F_T)$, and so $\pi_M(\mathcal{G}_n)$ is isomorphic to $\mathcal{G}_m$.

But the projections π_M, as M varies, form a uniform cover of projections – each coordinate is killed in the same number of π_M as each other one. It follows by Theorem 5.1 that $G(n) \geq G(m)^{4\binom{n}{2}/4\binom{m}{2}}$, as required. □

The same proof of course goes through in exactly the same way for k-SAT. Write $G_k(n)$ for the number of k-SAT functions on n variables, and put $G_k(n) = 2^{c_{k,n}\binom{n}{k}}$.

Theorem 5.3 *Let the constants $c_{k,n}$, $k \leq n$ be defined by*

$$G_k(n) = 2^{c_{k,n}\binom{n}{k}}.$$

Then for any k we have $c_{k,k} \geq c_{k,k+1} \geq \cdots$.

Proof. Identical to the proof of Theorem 5.2. □

Rather unexpectedly, for general k this gives a bound on $G_k(n)$ that is much better than the trivial bound, for large k. Note that all that is a priori obvious is that $G_k(n) \leq 2^{2^k\binom{n}{k}}$.

Theorem 5.4 *For any fixed k, we have $G_k(n) \leq 2^{\sqrt{\pi(k+1)}\binom{n}{k}}$ for $n \geq 2k$.*

Proof. Since the number of Boolean functions on n variables is 2^{2^n}, we have $G_k(2k) \leq 2^{2^{2k}} \leq 2^{\sqrt{\pi(k+1)}\binom{2k}{k}}$; in other words $c_{k,2k} \leq \sqrt{\pi(k+1)}$.

Thanks to Theorem 5.3, we can deduce that $c_{k,n} \leq \sqrt{\pi(k+1)}$ for all $n \geq 2k$, which is the required result. □

6 Open problems

The first question we would like to ask is whether something much stronger than Theorem 1.1 is true, namely that "almost every" 2-SAT function can be obtained from a monotone formula by relabelling the literals. In other words, we make the following conjecture.

Conjecture 6.1

$$G(n) = (1 + o(1))2^{n(n+1)/2}.$$

One would hope that the methods developed by Kleitman and Rothschild [12] and Promel and Steger [15] could be used to prove this conjecture, but so far we have been completely unsuccessful.

As with other statements of this flavour, a proof of this conjecture could be regarded as a first step towards a theory of random 2-SAT functions. It is perhaps worth contrasting this with the established theory of random 2-SAT *formulae*. If the Conjecture is true, it would imply that a typical 2-SAT function has a unique formula containing about $n^2/4$ clauses; on the other hand, a formula obtained by choosing even as few as $(1+\varepsilon)n$ clauses independently uniformly at random will almost surely give the trivial function. See Bollobás, Borgs, Chayes, Kim and Wilson [3] for a recent, very sharp, version of this result.

Secondly, it would be extremely interesting to estimate the number $G_k(n)$ of k-SAT functions on n variables, for general k or even for $k = 3$. Again, the number of monotone formulae, namely $2^{\binom{n}{k}}$, gives us a lower bound; and indeed we suspect the following.

Conjecture 6.2 *For any fixed k we have*

$$G_k(n) = 2^{(1+o(1))\binom{n}{k}}.$$

As remarked above, Theorem 5.2 implies that, if one can evaluate $G_k(n_0) = 2^{c\binom{n_0}{k}}$ for any particular value of n_0, then $G_k(n) \leq 2^{c\binom{n}{k}}$ for all $n \geq n_0$. However, we do not see how to use this to determine the precise asymptotic behaviour.

Needless to say, for general k, if this conjecture is true then we would go on to make the stronger conjecture (as for 2-SAT above) that almost every k-SAT function is a monotone one, or obtained from one by relabelling the literals.

A forthcoming paper of the first two authors discusses cases wth $k \geq n/2$, where the picture is totally different.

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