An Alternative to the Feltham Ohlson Valuation Framework

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Abstract

In this research we develop an alternative to the Feltham Ohlson investment valuation model. While the Feltham Ohlson model has no transparent role for management, in contrast, in our model management need to make optimal investment or divestment decisions on a period by period basis. We establish the functional relationship between accounting reports and optimal investment decisions. Our model provides both an alternative rationale for published accounting numbers having information content and an alternative framework for specifications of empirical testing of the value relevance of accounting data.

1 INTRODUCTION

The research is organised as follows. In the first subsection immediately below we discuss the established Feltham Ohlson valuation model and briefly review the main findings. We argue that the approach excludes consideration of important real options that typically arise empirically when making investment decisions. This is outlined in more detail in section two. In section three we introduce a new investment model in which real options naturally arise and can be solved for optimally. In the following section we then introduce the possibility of accounting based valuation. We consider how the optimal decision making arising from application of the the standard real options approach can be replicated by reference to appropriately defined accounting valuation routines. The new derived model of valuation based upon accounting numbers provides an alternative theoretical basis for testing the information content of published accounting numbers. We present concluding comments in section five.

1.1 The Feltham-Ohlson Model

There exists a number of review papers of the Feltham Ohlson (FO) approach such as Lo and Lys (1999) and Walker (1997) which thoroughly review the model and provide a critique of the approach. However, having critiqued the model these papers do not provide constructive alternative valuation approaches. Our focus instead is upon one of the central limitations of the FO approach; its lack of a well defined function for management with respect to project selection. In the following section we derive an alternative valuation model in which management have a role to play in (real options) project selection[1].

The FO model is normally developed by first considering a transformation of the traditional discounted future dividend valuation model:

$$W_t = \sum_{\tau=1}^{\infty} \gamma^{\tau} E_t(d_{t+\tau}). \tag{1}$$

Here W_t = market price of equity in the firm/project at date t, d_t = dividends paid at the end of each period t, $\gamma = (1+r)^{-1}$ the discount rate and E_t = the expectations operator. There are two natural interpretations of (1). The first has expectations computed using an equivalent martingale measure for the equity price (a modelling assumption that such exists) and then the discount rate r is interpreted as the riskless rate. Alternatively, if the returns on equity W_t are modelled as independently and identically distributed (i.i.d.; assuming such a belief on the part of investors), then the physical probability for the distribution of equity price may be used as an equivalent procedure, in which case the discount rate becomes the constant expected rate of return and that of necessity is set equal to the 'required rate of return' for the given class of risk. Our model is based on the latter premise; that is to say the model assumes that management control economic activities so that expected return is set

[1] To the best of our knowledge only one other author considers a similar modelling approach. Yee (2000) also incorporates project selection but as we shall see, his model is different in certain crucial ways.

equal to the 'required rate of return'. The precise significance of this rule is studied in later sections and involves recognition of embedded investment call and put options which are the hallmarks or the real options approach to investment. An objective for the analysis is to identify the risk class of a project and hence the required rate of return.

Equation (1) requires a technical assumption[2]. From this equation, and also subject to a similar kind of technicality[3], appealing to the clean surplus identity gives:

$$B_t = B_{t-1} + v_t - d_t \tag{2}$$

(where $B_t =$ book value of equity at t, $v_t =$ earnings at the end of period t) leads to the identity:

$$W_t = B_t + \sum_{\tau=1}^{\infty} \gamma^\tau E_t(\widetilde{v}_{t+\tau}), \tag{3}$$

where residual income, or 'abnormal earnings' as it is sometimes alternatively called, is defined by:

$$\widetilde{v}_t \equiv v_t - rB_{t-1}.$$

The most attractive feature of this approach is that it links valuation to observable accounting data. The ability to re-express (1) in a way that gives accounting center stage via (3) has been well known for a considerable time. Ohlson's particular contribution was to set out a specific proposal for how $\widetilde{v}_{t+\tau}$ came about. In particular he posited that:

$$\widetilde{v}_{t+1} = \omega\widetilde{v}_t + x_t + \varepsilon_{t+1} \tag{4}$$

where $0 \leq \omega$ and $x_t =$ value relevant information not yet captured by accounting and $\varepsilon_{t+1} =$ is a mean zero disturbance term. In turn he assumed:

$$x_{t+1} = gx_t + \eta_{t+1} \tag{5}$$

where $g < 1$ and η_{t+1} is a zero-mean disturbance term. Together (4) and (5) imply that abnormal earnings follow an AR(1) process. It is apparent immediatedly that the Ohlson approach assumes a black box model of management since nowhere does the Ohlson model consider project selection or opportunities. Similarly the FO extension which allows for conservative accruals is silent with respect to project opportunities (real options). Thus while the FO approach does establish a dependence of abnormal earnings on book value it does so via a simple (decision opaque) mechanistic formulation.

In principle this exogenous method used to establish a dependence may be satisfactory if the assumed dependence is empirically supported in a robust fashion. However, as has been argued by numerous authors including recently Yee (2000), there are concerns that it is not. At the heart of the various limitations of the FO approach is the lack of a role for management in decision making. We shall now establish this limitation more formally by comparing and contrasting the FO approach to

[2]The 'no bursting bubble' assumption $\gamma^\tau E[W_\tau] \to 0$ as $\tau \to \infty$ is required here.

[3]Namely: $\gamma^\tau B_\tau \to 0$ as $\tau \to \infty$, i.e. book value does not grow faster than the riskless or required rate of return (whichever is appropriate).

the general real options valuation model. This allows us to see directly that a major limitation of the FO approach is that it is essentially a static theory of investment in which once management make an investment they implicitly ignore the type of strategic new investments and divestments that typically characterize the rich empirical setting in which investment decisions are taken.

To summarise then, our objective is to develop an alternative model to one in which the dynamic path for $\widetilde{v}_{t+\tau}$ is imposed via a simple restrictive recursive assumption. Instead we wish to derive an optimally dynamic flexible decision rule which requires investment levels to be dynamically adjusted by management in response to time varying stochastic conditions. Put simply, we wish to develop a model in which management, responding to current period reported poor (good) residual accounting income may reduce (increase) investment. That is, we suggest that our model provides a more natural bridge to structure empirical observations in which firms routinely contract, shut down or expand investment projects. Critical to our approach is the ability to identify optimal dynamic investment strategies. Given the complexity of such problems in general, we shall adopt a simple stylised approach for capturing the investment environment in which firms operate. However, before presenting our formal model we shall next briefly review an influential real options paper which leads naturally to our model specification.

2 The Real Options Approach to Investment Valuation

We commence our discussion of the real options approach by first briefly reviewing the work of Abel, Dixit, Eberley and Pindyck (1996) which presents an easily accessible introduction to the literature and clearly demonstrates the above outlined limitation with the Feltham Ohlson approach. After setting out the ADEP model we discuss various extensions which lead in a natural way to the specification of our alternative model.

In a simple two period setting the model considers the problem of whether a firm should add to or reduce its opening (first period) stock of capital K_0 which is purchased at a unit price of b_0. This is to be determined given the following three complications: the future (period one) purchase price of capital b_H may exceed its current price (costly expandability; $b_H > b_0$); the future resale price of capital b_L may be less than its current price (costly reversibility; $b_L < b_0$) and finally second period revenues from employing capital are stochastic. The stochastic element is introduced as follows[4]. In the first period total revenue from installed capital is $r(K_0)$, however in the second period the revenue denoted $R(K, a)$, has a stochastic component determined by the realisation of a. Subsequently in the second period after a has been revealed the firm adjusts the capital stock to a new optimal level denoted $K_1(a)$. Differentiating the revenue function with respect to K the following two critical values of a are identified:

$$R_K(K_0, a_L) \equiv b_L \qquad \text{and} \qquad R_K(K_0, a_H) \equiv b_H$$

that is, the optimal (marginal) decision rule is:

- when $a < a_L$ it is optimal to sell capital to the point that $R_K(K_1, a) = b_L$
- when $a_L \leq a \leq a_H$ it is optimal to neither purchase nor sell capital, that is $K_1(a) = K_0$

[4]For brevity we are not including details of all the regularity conditions since they can be found in the original text.

- when $a > a_H$ it is optimal to purchase capital until $R_K(K_1, a) = b_H$;

and so the present value of net cash flows $V(K_0)$ accruing to the firm commencing with capital stock K_0 in period zero with inter period discount rate γ, is given by:

$$\begin{aligned} V(K_0) &= r(K_0) + \gamma \int_{-\infty}^{a_L} \{R(K_1(a), a) + b_L[K_0 - K_1(a)]\} dF(a) \\ &\quad + \gamma \int_{a_L}^{a_H} R(K_0, a) dF(a) + \gamma \int_{a_H}^{\infty} \{R(K_1(a), a) - b_H[K_1(a) - K_0]\} dF(a). \end{aligned} \tag{6}$$

Thus the period one decision faced by the firm is:

$$K_0 = \arg\max V(K_0) - b_0 K_0,$$

and the Net Present Value Rule can be interpreted from the first order condition as requiring:

$$\begin{aligned} V'(K_0) &\equiv r'(K_0) + \gamma b_L F(a_L) + \gamma \int_{a_L}^{a_H} R'(K_0, a) dF(a) + \gamma b_H[1 - F(a_H)] \\ &= b_0. \end{aligned} \tag{7}$$

This equates the period-one and onwards marginal return to capital to the initial marginal cost; note that the terms after $r'(K_0)$ which take into account the optimal change in capital stock in the following period. An alternative interpretation is also available. ADEP point out that equation (7) can be interpreted using Tobin's q-theory of the marginal value of capital. In this instance the marginal value of capital is

$$q \equiv V'(K_0),$$

and so the optimal investment rule can be identified by management if they determine q.

With respect to implementing this rule ADEP (p 761) comment that this (theoretically correct) rule can be difficult to aply in practice because "for a manager contemplating adding a unit of capital, it requires rational expectations of the path of the firm's marginal return to capital through the indefinite future" and thus in practice the most commonly used proxy for the correct NPV "treats the marginal unit of capital installed in period 1 as if the capital stock is not going to change again". In this case the marginal value of $V'(K_0)$ is approximated by:

$$\widetilde{V}'(K_0) \equiv r'(K_0) + \gamma \int_{-\infty}^{\infty} R_K(K_0, a) dF(a) \tag{8}$$

and ADEP describe this replacement for the left-hand side of (7) as yielding the *naive NPV rule*.

At this point it is very helpful to note that the difference between $\widetilde{V}'(K_0)$ and $V'(K_0)$ is given precisely by the embedded put and call options present in the problem. To see this we can rewrite (6) as:

$$\begin{aligned} V(K_1) &= r(K_1) + \gamma \int_{-\infty}^{\infty} R(K_0, a) dF(a) \\ &\quad + \gamma \int_{-\infty}^{a_L} \{[R(K_1(a), a) - b_L K_1(a)] - [R(K_0, a) - b_L K_0]\} dF(a) \\ &\quad + \gamma \int_{a_H}^{\infty} \{[R(K_1(a), a) - b_H K_1(a)] - [R(K_0, a) - b_H K_0]\} dF(a). \end{aligned} \tag{9}$$

or more succinctly as:

$$V(K_0) = \widetilde{V}(K_0) + \gamma P(K_0) - \gamma C(K_0) \tag{10}$$

where

$$\begin{aligned}
\widetilde{V}(K_0) &\equiv r(K_0) + \gamma \int_{-\infty}^{\infty} R(K_0, a) dF(a), \\
P(K_0) &\equiv \int_{-\infty}^{a_L} \{[R(K_1(a), a) - b_L K_1(a)] - [R(K_0, a) - b_L K_0]\} dF(a) \\
C(K_0) &\equiv \int_{a_H}^{\infty} \{-[R(K_1(a), a) - b_H K_1(a)] + [R(K_0, a) - b_H K_0]\} dF(a)
\end{aligned}$$

where $\widetilde{V}(K_0)$ is the expected present value over both periods keeping the capital stock fixed at K_0, i.e. not allowing expansion or contraction of the capital stock. Now

$$P'(K_0) = \int_{-\infty}^{a_L} \{b_L - R'(K_0, a)\} dF(a) = E[\max\{b_L - R'(K_0), 0\}]$$

is the value of a (marginal) put[5] on the marginal product of capital with exercise price b_L corresponding to selling back. Similarly $C'(K_0)$ is the value of a (marginal) call on the marginal product of capital with exercise price b_H:

$$C'(K_0) = \int_{-\infty}^{a_L} \{-b_H + R'(K_0, a)\} dF(a) = E[\max\{R'(K_0) - b_H, 0\}]$$

Thus, given (7), to capture the incentives to invest and divest we can decompose the marginal value into three components:

$$q = V'(K_0) = \widetilde{V}'(K_0) + \gamma P'(K_0) - \gamma C'(K_0).$$

Notice that the present value of expansion requires additional outlay (hence the negative term), whereas contraction generates additional income (hence the positive term)

To summarise in the first period optimality requires management to choose K_0 so that

$$\widetilde{V}'(K_0) = b_0 - \gamma P'(K_0) + \gamma C'(K_0). \tag{11}$$

That is under the *naive rule* in which management set $\widetilde{V}'(K_0) = b_0$, management are ignoring (strategic) options values to contract or expand in the second period and hence typically would choose K_1 suboptimally.

[5] The put corresponds to the option to reduce the capital stock K_1 by selling k of the existing stock at b_L whenever $a < a_L$. Thus the realized value of the firm when the realization a is below a_L is to first order

$$\begin{aligned}
& r(K_1) + \gamma(R(K_1 - k) - R(K_1) + b_L k) \\
= \; & r(K_1) + \gamma k(b_L - R'(K_1)).
\end{aligned}$$

Moreover it is straight-forward to show[6] that the FO model is an implementation of the naive investment rule which ignores the options to expand and contract available in most real options settings and hence accounting valuation theory based upon that approach is unlikely to be able to capture how accounting valuation impinges upon actual firm dynamic investment strategy including both expansion and contraction possibilites.

The objective of the next section is to develop a simple model which overcomes this deficiency in that management formally need to evaluate options to expand and contract each period and moreover it extends the two period ADEP model to more realistic investment horizons of $N > 2$ finite periods[7]. After setting out the revised finite horizon investment model we then return to consider accounting valuation issues in the following section.

3 Optimal Investment by Management Allowing Contraction and Expansion

Our model specification is somewhat different from that of ADEP. At the heart of the difference is the way in which capital is utilised, in particular we develop a model of (installed) capital in which capital depreciates through use (as directed by management) rather than at a constant rate or not at all. We make this assumption to allow for the possibility that the net book value of an investment asset after subtracting accumulated depreciation could in principle be equal to the economic value of the asset to the organization. In contrast in the ADEP framework the asset is assumed never to depreciate. In addition we extend the investment planning horizon beyond a simple two period framework to a general finite horizon setting. In order to introduce the difference in specification as transparently as possible we first consider a two period model variant of the ADEP model.

3.1 The two period model

In general firm investment is subject to multiple sources of uncertainty. In the ADEP model the source of uncertainty is the price of finished output. In contrast in our model we focus upon the input price of capital as the principle source of uncertainty[8]. Our objective here will be to characterise $V(K_0)$ the optimal value function for capital usage. As we shall see by making certain functional assumptions for the operating environment we will be able to go further than ADEP since not only can we identify equivalent optimality conditions to (10) but moreover we can solve for the conditions once we have derived the funtional form for the optimal value function $V(K_0)$.

[6]See Lo and Lys (2000). The FO approach simply assumes constant expansion (such as in the Gordan growth model) rather than period by period expansion or contraction as will be allowed for in the model developed below.

[7]This is not the only difference between the two models. As we shall see in the following section there are a number of other differences the most significant perhaps being that in our model setting depreciation occurs through use rather than at a constant rate or alternatively not at all as in the ADEP model.

[8]A generalised version of our model in which both the input price and the output price are stochastic is available from the authors. The two sources of uncertainty complicates the analysis by requiring consideration be centered around the ratio of output to the input price without changing the general nature of results substantitively.

We now develop our model via direct comparison to the ADEP approach. Simplifying the output return side[9] we take the time $t = 0$ revenue to be $r(K) = 2\sqrt{K}$ and the time $t = 1$ revenue to be $R(K, a_1) = 2a_1\sqrt{K}$ where $a_1 > 0$ represents the unit sale price of the output at time $t = 1$. To further simplify the analysis since in our model the the input price is the prime source of uncertainty we shall take $a_1 = 1$. Concentrating upon the source of uncertainty we shall allow b_1 the input purchase price of capital at time $t = 1$ (corresponding to the constant b_H considered by ADEP) to be stochastic. In addition, we assume that the resale price of the input is $b_1\phi_1$ at time $t = 1$ (instead of b_L in ADEP notation) where the discount factor $\phi_1 < 1$ reflects the partial irreversibility of earlier investment. For clarity of exposition ϕ_1 is deterministic in this model, but the model can be adapted to allow ϕ_1 to be stochastic. The fractional value of ϕ_1 is assumed to result from the input not being freely tradeable and this creates a fundamental incompletenesses in the specialist capital input market. This has important implications for the valuation of the firm; the assumptions of the standard martingale approach in real option theory posit the existence of a 'traded twin security' perfectly correlated with the real asset. In our case the real asset is the additional capital for which the purchase and sale prices diverge at time $t = 1$ by the factor ϕ_1 so that it is no longer possible to hold long and short positions at one price. We therefore abandon the simple martingale approach[10] and instead adopt the standard 'private values' dynamic programming approach[11] for valuation using the physical distribution of the input price b_1. This is the approach also taken by Abel and Eberley (1995) in their continuous-time infinite horizon model.

Commencing at time t_0, we assume that a firm has $u_0 = u_{t_0}$ ($u_{t_0} \geq 0$) units of capital in stock[12]. Given the firm can purchase some more capital in the next period the decision of how to allocate capital stock optimally between the current and latter period will ceteris paribus be driven by the capital input price process. We shall denote the one-period discounted[13] price of capital by b_t.

Although in general we use a sequence of times and corresponding prices that evolve geometrically, the price is nevertheless presented as though it evolves continuously as a geometric Brownian motion. Such an approach is dictated purely by mathematical convenience; the mathematics of optimisation is much streamlined by the assumption that at each time, price is distributed continuously rather than multinomially; the presence of interperiod prices is not referred to in any way because we have periodic management decision making. The price b_t has positive drift[14] (anticipated growth)

[9] In general we need not restrict attention to a square root formulation, all we need is concavity. The role of the square root specification is to maximize the simplicity of the presentation.

[10] A related situation is that of a four state model in which prices of a traded asset move up or down and an investor receives a preference shock to buy or sell. This single risky asset model is evidently incomplete and at best presents two martingales one for expansion and one for contraction if the appropriate buy:sell margin ratio is interpreted as a discount.

[11] See Dixit and Pindyck (199) for an extended discusssion of this point.

[12] Note u_0 in our notation corresponds to K_0 in ADEP notation. We do not adopt their notation because of the different way in which capital is "consumed" in the two models.

[13] By one-period-discounted we mean that if the asset is purchased for $p_n = p_{t_n}$ at the commencement of the time interval $[t_n, t_{n+1})$, then the unit opportunity cost of funds tied up in the asset are $p_n(1 + r) = b_n$ where b_n stands for b_{t_n} and r is the one-period interest rate. Alternatively one can regard the supplier as asking a price payable at the end of the period upon delivery.

[14] The drift is net of an implicitly assumed constant interest rate. Thus b_t is to be regarded as a depreciated price.

$\mu_b > 0$, and is presented in the traditional stochastic differential form:

$$db_t = b_t(\mu_b dt + \sigma_b dW_b(t))$$

where $W_b(t)$ is a standard Wiener process. For $t > s$, we let $Q(b_t|b_s)$ denote the (log-normal) cummulative distribution of b_t given b_s and we also let $Q_n(b) = Q(b_{t_n}|b_{t_{n-1}} = 1)$ denote the (log-normal) cummulative distribution of $b_n = b_{t_n}$ given that $b_{t_{n-1}} = 1$. When the context permits we drop the subscript n.

In the simplest model the manager observes the price at discrete times, in this case at times t_0 and t_1 and can purchase/resell capital at these discrete moments which we shall denote $z_0 = z_{t_0}$ and $z_1 = z_{t_1}$. In order to track the stock of capital carried forward between periods we shall denote the period t_0 opening capital stock as v_{t_0}, or just v_0, and closing stock as u_{t_0}, or just u_0. Let us now consider how to determine the optimal amount of capital u_{t_0} to carry forward to the next period given the amount purchased in the period is unrestricted, that is $z_{t_0} \geq 0$.

The manager now needs to maximize over both z_0 (≥ 0) and x_0 the profit[15]:

$$2\sqrt{x_0} - b_0 z_0 + \gamma V_0(v_0 + z_0 - x_0, b_0).$$

Here $V_0(u_0, b_0)$ denotes the future expected value given the current price b_0 and the capital stock carried forward u_0 paid for in a previous period. Equivalently, letting $u_0 = v_0 + z_0 - x_0$ we maximize:

$$2\sqrt{x_0} - b_0(u_0 + x_0 - v_0) + \gamma V_0(u_0, b_0). \tag{12}$$

Then when choosing optimally the closing stock of capital u_0 the first order condition from (12) gives:

$$\gamma V_0'(u_0, b_0) = b_0 \tag{13}$$

and[16]:

$$x_0 = \frac{1}{b_0^2}, \tag{14}$$

where the prime denotes differentiation with respect to u. Note that (13) implies that for investment u_0 to be chosen optimally the unit return need be equated to the constant return $1 + r$, namely

$$\frac{V_0'(u_0, b_0)}{b_0} = \gamma^{-1} = (1 + r).$$

A firm planning to divest, i.e. taking $z_0 < 0$ faces a similar problem. If the resale discount is ϕ_0 the firm considers the corresponding problem: maximize over both z_0 (< 0) and x_0 the profit[17] :

$$2\sqrt{x_0} - \phi_0 b_0 z_0 + \gamma V_0(v_0 + z_0 - x_0, b_0),$$

[15]That is choice of the variables to maximise the sum of current profit plus the optimal value function $V(.)$ reflecting future optimal period payoffs.

[16]This very simple nature of this result is why we utilise the square root specification.

[17]That $\phi < 1$ is standard in the literature, otherwise if $\phi = 1$ we would have the possibility of simple portfolio rebalancing.

or equivalently letting $u_0 = v_0 + z_0 - x_0$,

$$2\sqrt{x_0} - \phi_0 b_0(u_0 + x_0 - v_0) + \gamma V_0(u_0, b_0).$$

Thus the first order condition for u_0 is:

$$\gamma V_0'(u, b_0) = \phi_0 b_0 \tag{15}$$

and for x_0 is:

$$x_0 = \frac{1}{\phi_0^2 b_0^2}. \tag{16}$$

We return to the investment version and let $u = \widehat{u}(b_0)$ denote the solution to equation (13).

Remark (Identification of the two period optimal value function):
We note that in our two-period model we have:

$$\begin{aligned} V_0(u|\phi_1, b_0) &= \int_0^{\widetilde{b}_L} (\frac{1}{b_1} + b_1 u) dQ(b_1|b_0) + 2\sqrt{u} \int_{\widetilde{b}_L}^{\widetilde{b}_L/\phi_1} dQ(b_1|b_0) \\ &\quad + \int_{\widetilde{b}_L/\phi_1}^{\infty} (\frac{1}{\phi_1 b_1} + \phi_1 b_1 u) dQ(b_1|b_0) \end{aligned} \tag{17}$$

where $\widetilde{b}_L = 1/\sqrt{u}$. The three integrals classify investment by the corresponding three input price policy ranges[18] according to the ranges of integration, as follows:

(**U**) The under-invested range ($0 \leq b_1 \leq \widetilde{b}_1$), in which additional investment in capital is made. Here as in (12) one maximises over $x_1 \geq 0$ the second period profit

$$2\sqrt{x_1} - b_1 x_1$$

with required input $x_1 = 1/b_1^2$ made available thorugh the purchase of $x_1 - u$ at a price b_1 and net revenue $2/b_1 - b_1(x_1 - u) = 1/b_1 + b_1 u$. Clearly the extreme case is zero purchase when $u = 1/b_1^2$ whence the limit $\widetilde{b}_L$.
(**IO**) The (endogenously) irreversible[19] over-invested range ($\widetilde{b}_L \leq b_1 \leq \widetilde{b}_L/\phi_1$), where all remaining capital (excess from period 0) is put into production.
(**RO**) The reversible investment range ($\widetilde{b}_1/\phi_1 \leq b_1 \leq \infty$), where some excess capital is resold. Here one maximise over $x_1 \leq u$ the second period profit

$$2\sqrt{x_1} + \phi_1 b_1(u - x_1)$$

[18] Equivalent to the three output price ranges in the ADEP model.
[19] Endogenous in the sense that though reversal is possible, it is never optimal in this setting to choose it and hence the firm acts as if the situation was irreversible.

Figure 1:

obtained by reselling an amount $u - x_1$ of the capital stock. The required input is $x_1 = 1/(\phi_1 b_1)^2$ yielding net revenue $2/(\phi_1 b_1) + \phi_1 b_1 (u - x_1) = 1/(\phi_1 b_1) + \phi_1 b_1 u$. The extreme case is $u = 1/(\phi_1 b_1)^2$ giving $b_1 = 1/(\sqrt{u}\phi_1)$. Thus:

$$V_0'(v, b_0, \phi_1) = \int_0^{\widetilde{b}_L} b_1 dQ(b_1|b_0) + \widetilde{b}_L \int_{\widetilde{b}_L}^{\widetilde{b}_L/\phi_1} dQ(b_1|b_0) + \int_{\widetilde{b}_L/\phi_1}^{\infty} \phi_1 b_1 dQ(b_1|b_0).$$

Again an alternative interpretation is possible with reference to Tobins q. Consider a policy of investment triggered by input prices below a threshold level of B. The average marginal benefit of such a strategy corresponds to the value of Tobin's marginal q. This we may compute from the last formula by writing B in place of $\widetilde{b}_L$ obtaining the function:

$$q(B, b_0) = \int_0^{B} b_1 dQ(b_1|b_0) + B \int_{B}^{\psi B} dQ(b_1|b_0) + \int_{\psi B}^{\infty} \phi_1 b_1 dQ(b_1|b_0).$$

At this point it is important to note that our assumption of a Cobb-Douglas type technology gives rise to the following homogeneity property

$$q(B, b_0) = b_0 q(B/b_0, 1)$$

which we will then apply. An inductive argument shows that this homogeneity property extends to all periods in the context of a Cobb-Douglas production function. The behaviour of $q(B, 1)$ is indicated in Figure 1 (that too is characteristic for multiple period models).

The importance of this function stems from the induced decomposition of the solution of (13) into two steps. The first step is to solve for $\widetilde{b}_1$

$$q(\widetilde{b}_1, b_0) = b_0. \tag{18}$$

and then solving $\widetilde{b}_1 = \widetilde{b}_L = 1/\sqrt{u}$ for u to obtain

$$\widehat{u}(b_0) = 1/(\widetilde{b}_1)^2.$$

We call (18) the **censor equation**. The solution exists and is unique if and only if $b_0 < E[b_1|b_0]$. i.e. provided prices are expected to rise. We call the value of b_1 given by $\widetilde{b}_1$ above i.e. solving (18) the **censor**.The homogeneity property of the censor implies that the censor is linear in b_0 so we have for a constant $\widehat{g}_1$ (the input price persistence factor)

$$\widetilde{b}_1(b_0) = \widehat{g}_1 b_0 \tag{19}$$

Thus

$$\widehat{u}(b_0) = \frac{1}{(\widehat{g}_1 b_0)^2}. \tag{20}$$

The corresponding problem for divestment calls for the solution of

$$q(\widetilde{b}_\phi, b_0) = \phi_0 b_0$$

or equivalently

$$q(\widetilde{b}_\phi, 1) = \phi_0 \tag{21}$$

and this will have a solution if and only if

$$\phi_0 > \inf_B q(B, 1).$$

The intuition is simple: if there is no solution, then there is no resale possible in that period.

Comparing (6) and (17), we can make the same re-arrangement as ADEP to give:

$$\begin{aligned} V_0(u, \phi, b_0) &= 2\sqrt{x(b_0)} + \gamma[2\sqrt{u}\int_0^\infty dQ(b_1|b_0) \\ &\quad + \int_0^{\widetilde{b}_L} (\frac{1}{b_1} + b_1 u - 2\sqrt{u}) dQ(b_1|b_0) + \int_{\widetilde{b}_L/\phi_1}^\infty (\frac{1}{\phi_1 b_1} + \phi_1 b_1 u - 2\sqrt{u}) dQ(b_1|b_0)] \end{aligned}$$

thus re-defining their notation rather than introducing new notation (since we will not use their representation again), we have similarly to (10)

$$V_0(u) = \widetilde{V}_0(u|b_0) - \gamma P(u, \widetilde{b}_1|b_0) + \gamma C(u, \widetilde{b}_1|b_0)$$

where

$$\begin{aligned} \widetilde{V}_0(u|b_0) &\equiv 2\sqrt{x(b_0)} + \gamma 2\sqrt{u}\int_0^\infty dQ(b_1|b_0), \\ P(u, \widetilde{b}_1|b_0) &\equiv \int_0^{\widetilde{b}_L} 2\sqrt{u} - (\frac{1}{b_1} + b_1 u) dQ(b_1|b_0) \\ C(u, \widetilde{b}_1|b_0) &\equiv \int_{\widetilde{b}_L/\phi}^\infty (\frac{1}{\phi_1 b_1} + \phi_1 b_1 u - 2\sqrt{u}) dQ(b_1|b_0) \end{aligned}$$

with $\widetilde{b}_L = 1/\sqrt{u}$ and where as before $\widetilde{V}_0(\widetilde{b}_1)$ is the expected present value over both periods keeping the capital stock carried forward fixed at u. Note that in view of the reciprocal relation between a and b the put and call have switched roles vis a vis ADEP.

As before looking at the first order conditions[20] we have, writing $\widetilde{b}_1$ for $\widetilde{b}_L$:

$$\begin{aligned}
V_0'(u|\phi_1, b_0) &= \int_0^{\widetilde{b}_1} b_1 dQ(b_1|b_0) + \widetilde{b}_1 \int_{\widetilde{b}_1}^{\widetilde{b}_1/\phi_1} dQ(b_1|b_0) + \\
&\quad +\phi_1 \int_{\widetilde{b}_1/\phi_1}^{\infty} b_1 dQ(b_1|b_0) \\
&= \widetilde{b}_1 \int_0^{\infty} dQ(b_1|b_0) - \int_0^{\widetilde{b}_1} (\widetilde{b}_1 - b_1) dQ(b_1|b_0) \\
&\quad +\phi_1 \int_{\widetilde{b}_1/\phi_1}^{\infty} (b_1 - \widetilde{b}_1/\phi_1) dQ(b_1|b_0) \\
&= \widetilde{b}_1 - E[\max(\widetilde{b}_1 - b_1, 0)] + E[\max(\phi_1 b_1 - \widetilde{b}_1, 0)] \qquad (22) \\
&= \widetilde{V}_0'(\widetilde{b}_1|b_0)/\gamma - P'(\widetilde{b}_1|b_0) + C'(\widetilde{b}_1|b_0) \\
&= q.
\end{aligned}$$

Comparison of (22) and (11) yields the key insight that the firm should evaluate the embedded investment call and put options with strike price given by the censor. In this respect the censor $\widetilde{b}_1$ determines the effective 'future' unit price (effective expected next period price) of inputs and thus delivery at that price requires the planner to: (i) receive compensation against that price for surrender of expansion potential and (ii) pay additionally to that price a compensation for enjoyment of contraction potential[21].

[20] With due consideration for Leibniz Rule.

[21] Alternative interpretation: The naive non-linear view is that one unit of capital next period will be worth $\widetilde{b}_1$ and leads to an inventory of $G(\widetilde{b}_1)$ but the marginal valuation ignores the present value of the option to expand when it is cheap to do so (i.e. $b_1 < \widetilde{b}_1$) and this will call for extra outlay (hence the negative sign of this PV) and also ignores the option to contract when $b_1 > \widetilde{b}_1/\phi$ so that it is worth selling for ϕb_1which brings in extra income. It is possible to use put-call symmetry (parity) to obtain

$$\begin{aligned}
F_1'(u, \phi, b_0) &= \int_0^{\widetilde{b}_1} b_1 q(b_1|b_0) db_1 + \widetilde{b}_1 \int_{\widetilde{b}_1}^{\widetilde{b}_1/\phi} q(b_1|b_0) db_1 + \\
&\quad +\phi \int_{\widetilde{b}_1/\phi}^{\infty} b_1 q(b_1|b_0) db_1 \\
&= E[b_1] - \int_{\widetilde{b}_1}^{\widetilde{b}_1/\phi} (b_1 - \widetilde{b}_1) q(b_1|b_0) db_1 + \\
&\quad -(1-\phi) \int_{\widetilde{b}_1/\phi}^{\infty} b_1 q(b_1|b_0) db_1
\end{aligned}$$

This may be interpreted as comprising first: the naive expected value of holding one unit of stock, short a limited call (operable in a limited range) and $(1-\phi)$ units short of an asset-or-nothing option..

Remark: *The optimal investment rule is determined by evaluating the optimal investment or divestment such that the marginal benefit of capital (q) is equal to the naive NPV together with the value of the marginal (short) put and (long) call options which have a strike price given by the optimally chosen censor.*

3.2 Finite horizon ($n > 2$): Identification of $V_n(.)$

We now adopt the following notational assumption in aid of reducing subscript use. If at the end of period $t-1$ we have u_{t-1} capital stock left over for the commencement of production in period t, we denote the capital stock at commencement of new production by:

$$u_{t-1} = v_t.$$

Then when the period of analysis is unambiguous we can thus drop the time subscript and simply refer to opening stock v and closing stock u for the period under consideration. Applying this simplified notation the following general characterization is then possible: for each n there exists a 'capital carry-forward function' $u(v, b)$, and a input price censor function $b(v)$ and a constant ψ such that:

$$\begin{aligned} V_n'(v, b_n, \phi_{n+1}) &= \int_0^{b(v)} b_{n+1} dQ(b_{n+1}|b_n) + \int_{b(v)}^{\psi b(v)} (v - u(v, b_{n+1}))^{-1/2} dQ(b_{n+1}|b_n) \\ &\quad + \int_{\psi b(u)}^{\infty} \phi_{n+1} b_{n+1} dQ(b_{n+1}|b_n). \end{aligned}$$

Assuming a general concave revenue function $f(x)$ in place of the square-root form, the presence of an additional period of production moves the exercise price (trigger) down. Here is the intuition: the provision for the future is the greater the further the horizon, but the trigger varies inversely with quantity so the the trigger is smaller the further the horizon; at the same time the manager is less likely to sell stock back at a discount if he has the option to use that same stock at a later date. The general formula, though daunting, is not much different[22]:

$$\begin{aligned} V_{n-1}'(u_n|\phi_n, b_{n-1}) &= \widetilde{b}_n - E[\max(\widetilde{b}_n - b_n, 0)] + E[\max(\phi_n b_n - \widetilde{b}_n, 0)] \\ &\quad + \int_{\widetilde{b}_n}^{\widetilde{b}_n/\phi_n} (f'(x_n(u_n, b_n)) - \widetilde{b}_n) dQ(b_n|b_{n-1}) \\ &\quad + \int_{\widetilde{b}_n/\phi_n}^{h_n(\widetilde{b}_n, \phi_n)} (f'(x_n(u_n, b_n)) - \widetilde{b}_n) dQ(b_n|b_{n-1}), \end{aligned}$$

where the first line refers to a strategy of not carrying forward capital. Here $\widetilde{b}_n = b_n(u, 1)$ and $x_n(u, b_n)$ is the optimal demand in period n for input given a stock u of input and current input price of b_n.

[22]The form of the optimal solution changes as we change the number of periods as follows. As we increase the number of periods this increases the range of inactivity since with more periods (to act on the volatility) the chance of eventually experiencing sufficiently good demand conditions to use up existing "excess" stock increases and hence the benefit of selling it at a discount commensurately reduces.

Generalising the two period model we can then show[23] that the future expected value is given by a formula incorporating three expected values according as the firm uses one of its three options: investment, divestment or mere partitioning of its capital stock between current and future use. The exact form of the formula is (see Appendix A)

$$
\begin{aligned}
&V_{n-1}(v, b_{n-1}) = \\
&\int_0^{b_n(v,1)} f(G(b_n)) - b_n G(b_n) + b_n(v - \widehat{u}_n(1, b_n)) + V_n(\widehat{u}_n(1, b_n), b_n) dQ_n(b_n) \\
&+ \int_{b(v,1)}^{b(v,\phi)} f(v - u_n(v, b_n)) + V_n(u_n(v, b_n), b_n) dQ_n(b_n) \\
&+ \int_{b(v,\phi)}^{\infty} f(G(\phi_n b_n)) - \phi_n b_n G(\phi_n b_n) + \phi_n b_n(v - \widehat{u}_n(\phi_n, b_n)) + V_n(\widehat{u}_n(\phi_n, b_n), b_n) dQ_n(b_n),
\end{aligned}
$$

where v is the opening stock, ϕ_n the discount factor for the next period, $\widehat{u}_n(1, b_n)$ is the carry-forward into a following period when investing, $\widehat{u}_n(\phi_n, b_n)$ is the carry-forward when divesting and $u_n(v, b_n)$ is the carry-forward in the absence of investment or divestment. Under the integral signs we see period n production income, future costs of additional investment or future income from divestments, all these granted v is treated as having sunk cost[24]. It is the first two terms on the right, namely $f(G(b_n)) - b_n G(b_n)$, which merit particular attention. Here $G(b_n) = (f')^{-1}(b_n)$ is an internal optimal demand for input maximising $f(x) - b_n x$ over x; let us denote it temporarily by x_n. Since $b_n = f'(G(b_n)) = f'(x_n)$ we see that the indirect profit $f(G(b_n)) - b_n G(b_n)$ can also be written as $f^{\#}(x_n)$, where $f^{\#}(x) = f(x) - xf'(x)$. An inductive application of the recurrence formula coupled with some re-arrangements of the other terms yields the following identity in terms of indirect profits for the undiscounted future value of the project given a current capital stock u_n :

$$
V_n(u_n | b_n) = q_n u_{n+1} + E[\sum_{m=n+1}^{N} \gamma^{m-n-1} f^{\#}(x_m^*)], \tag{23}
$$

that is, on the right-hand side we sum the existing carry-forward capital stock u_{n+1} evaluated at Tobin's q, plus the sum of all future indirect profits where:
i) $f^{\#}(x) = f(x) - xf'(x)$ denotes the indirect profit function associated with the production function $f(x)$;
ii) $u_{m+1} = u_{m+1}(u_m | b_n, ..., b_m)$ is the optimal carry-forward from period m to period $m + 1$ given the price history $b_n, ..., b_m$;
iii) $x_m^* = x_m^*(u_{m-1}, b_m)$ is the general optimal demand for input at time m (so that in certain circumstances $x_m^* = x_m$);
iv) $q_m = q_m(u_m, b_m)$ is the period-m Tobin's marginal q, defined as the average marginal benefit of utilization of a unit of input in period m given the current value of b_m and the *opening* stock u_m of the current period. Here q is characterized along the lines of ADEP as being composed of:

[23] Technical details are avaialbale from the authors upon request.
[24] Thus the total value of the firm in time t_n money must add to the given formula past income and the historic cost of v suitably compounded.

- a certainty equivalent price less the put option to expand plus the call option to contract plus the option to carry forward unused stock, i.e. typically is of the form:

$$\begin{aligned} q_0 &= \widetilde{b}_1 - E[\max(\widetilde{b}_1 - b_1, 0)] + E[\max(\phi_1 b_1 - \widetilde{b}_1, 0)] \\ &\quad + \int_{\widetilde{b}_1}^{h(\widetilde{b}_1,\phi)} (f'(x_1(u, b_1)) - \widetilde{b}_1) dQ(b_1|b_0). \end{aligned}$$

Note that the value of the firm measured in time t_n values and 'ignoring the past' is

$$f^{\#}(x_n^*) + \gamma V_n = f^{\#}(x_n^*) + \gamma q_n u_{n+1} + E[\sum_{m=n+1}^{N} \gamma^{m-n} f^{\#}(x_m^*)]$$

To summarise we can generalise the model to multiple periods and we can identify after taking appropriate discounting the form of the optimal value function as taking the specific form:
- q adjusted value of the opening capital stock plus the sum of expected indirect incomes
At this stage we could potentially still be lost how to implement this rule because we may not know how to predict expected indirect income. However we have established that given a Cobb-Douglas technology or more specifically square root functionality:
- the period n indirect profit function takes a notionally simple form; it is $1/b_n$ when the project is under-invested, $1/(\phi_n b_n)$ when it is over-invested, and an intermendiate value in the third regime.
Thus we can identify $V_n(u_n|b_n)$ by forming expectations over the input price process b_n. Furthermore forming this expectation simply requires looking at the appropriately censored sum of the current input price weighted by the input price persistence factors g_i, the generalised version[25] of (19).

Having now identified the optimal value function which guides mangement adjusting investment behavior in response to time varying stochastic capital input price changes we now turn to consider the implications for the usefulness of accounting based values such as residual income in guiding optimal managerial decision making.

4 Accounting Based Investment Decision Making

In order to illustrate the link between our results and that of the FO model we shall commence our discussion by considering a special case of the investment model we developed in the previous section. In this section our principle concern is to consider the possible justification for management basing investment decisions on the level of accounting residual income. In their model FO posited a linear model relationship. In contrast in our endogenous model setting we shall show that linearity does not hold with generality. However, in order to aid development of intuition we first consider in the following subsection one special case in which our model aproximately agrees with the FO assumption. In the following subsections we then demonstrate the lack of generality of the FO linear model.

[25]We will give a specific example in the next section below.

This special case where the two models approximately coincide occurs when there is no stock of capital in place and it is always optimal for management to increase the capital stock. As we shall see shortly the empirical implications of our model will depend crucially upon what sort of investment regime we are in, capital contraction, maintenance, or expansion. When testing the model a further consideration will be the level of data aggregation, that is whether the current input purchase price for investment is observed or only the physical stock level and its previous cost. We will develop our analysis in each of the two situations.

Another important factor in our analysis is that we prefer to use an accounting valuation convention based on current value costs rather than historic costs. We explain in Appendix F that this leads to a differently constructed residual income stream whose discounted sum, displayed in (23), is identical to that based on historic cost. It is however more tractable for our analysis.

4.1 Regime (i): Under-invested in capital stock

In the multi-period setting, suppose first that at the start of business, there is no stock of capital. Via a generalisation of (19) and (20) it can be shown that solving the first order condition for the optimal value function to derive the optimal capital purchase corresponds to requiring that a capital stock be purchased of:

$$\widehat{v}_0 = \frac{1}{b_0^2} + \frac{1}{(\widetilde{g}_{1,1} * b_0)^2} + ... + \frac{1}{(\widetilde{g}_{1,N} * b_0)^2}, \tag{24}$$

where

$$\widetilde{g}_{n,m} = \widehat{g}_n \cdot \widehat{g}_{n+1} \cdot ... \cdot \widehat{g}_m$$

and $\widehat{g}_1, ..., \widehat{g}_N$ are the period by period price input persistence parameters of the model, acting like Gordon growth factors associated with the individual stages of the project. Indeed

$$\begin{aligned}
\widehat{v}_0 &= \frac{1}{b_0^2} + \widehat{v}_1(b_0 g_1) \\
&= \frac{1}{b_0^2} + \frac{1}{b_0^2 g_1^2} + \widehat{v}_2(b_0 g_1 g_2)... \\
&= \frac{1}{b_0^2} + \frac{1}{b_0^2 g_1^2} + \frac{1}{b_0^2 g_1^2 g_2^2} + ...
\end{aligned}$$

Thus the stock is built up as though the prices in the future were known to be $b_m = \widehat{g}_1 \cdot \widehat{g}_2 \cdot ... \cdot \widehat{g}_m b_0$.

In general at time t_i the optimal opening investment stock to purchase is given by:

$$\widehat{v}_i(b_i) = \frac{1}{b_i^2} + \widehat{u}_i(b_i) = \frac{1}{b_i^2} + \frac{1}{(\widetilde{g}_{i+1,i+1} * b_i)^2} + ... + \frac{1}{(\widetilde{g}_{i+1,N} * b_i)^2}. \tag{25}$$

The significance of this result is that we can recover a valid form of the FO model under regime (i) when opening stock $v_i < \widehat{v}_i$. Indeed the current optimal accounting profit is given by $1/b_i$ and the optimal expected future residual income may be obtained by increasing the stock to $\widehat{v}_i = \widehat{v}(b_i)$ and the value of that optimal expected future residual income is computed from the following formula derived in Appendix B:

$$\overline{F}(\widehat{v}(b_i), b_i) = \frac{1}{b_i}\overline{F}(\widehat{v}(1), 1).$$

Notationally we may write this in the form:

$$\widehat{V}_i = \frac{C_{i;N}}{b_i} \tag{26}$$

for some constant $C_{i;N}$. That is the optimal future value of the firm is given by the current accounting profit multiplied by a certain constant (which is dependent on volatility).

Since management find themselves with insufficient opening capital stock $v_i < \widehat{v}_i$, they respond at time t_i by:

- making an additional purchase of $\widehat{v}_i - v_i$ units of capital at a price b_i

- allocating $1/b_i^2$ to current production thus producing a residual income *component* $y_i = 1/b_i$;

and thus management estimates the optimal expected future residual income from the project is of the form

$$\widehat{V}_i = \frac{C_{i;N}}{b_i} = C_{i;N} y_i$$

which is linear in a component of the current period residual income as in the FO model.

The cash payment of $b_i[\widehat{v}_i - v_i]$ is a further component of the full residual income which we denote Y_i and which is:

$$\begin{aligned} Y_i &= \frac{2}{b_i} - [\frac{\widehat{v}_i(1)}{b_i^2} - v_i]b_i \\ &= \frac{1}{b_i} - [\frac{\widehat{v}_{i+1}(1)}{b_i^2} - v_i]b_i = \frac{[1 - \widehat{v}_{i+1}(1)]}{b_i} + v_i b_i. \end{aligned}$$

Notice that since v_i is assumed less than $\widehat{v}_i$ we have $Y_i \leq 2/b_i$; the minimum of Y_i relative to b_i occurs when $(b_i)^2 = [1 - \widehat{v}_{i+1}(1)]/v_i$ i.e. $v_i = \widehat{v}_i(b_i)$.

Our analysis now splits into two cases according as the value of v_i, or alternatively the value of b_i, is empirically observed. In the first case we establish the relationship between future value V_i and residual income Y_i by eliminating the unobserved variable b_i. We obtain the relation that

$$Y_i = \frac{\widehat{V}_i}{C_{i;N}}[1 - \widehat{v}_{i+1}(1)] + v_i \frac{C_{i;N}}{\widehat{V}_i}. \tag{27}$$

This relation holds only for $Y_i \leq 2/b_i$ and we need to consider the complementary interval in order to specify the relation fully (and that will be obtained in the following subsections). For now notice that the formula derived is hyperbolic, and so inverting it we see only the asymptotic linearity:

$$V_i \approx \frac{C_{i;N}}{[1 - \widehat{v}_i(1)]} Y_i$$

in play when the project is under-invested. Note that the slope of this line depends on volatility. Furthermore note that in general $C_{i;N} \neq C_{i+1;N}$ and so the linear coefficient will be time varying.

In the second case we need only note the self-evident linear relation between the total value and the residual income which is

$$\begin{aligned} TV &= V_i + Y_i \\ &= \frac{C_{i;N}}{b_i} + Y_i \end{aligned} \tag{28}$$

with the parameters b_i and $C_{i;N}$ known. We return to this formula below.

Having said our model is in partial agreement with the FO model when the firm is expanding its investment we shall now show that under the two other regimes our model contrasts sharply with that of FO.

4.2 Regime (ii): Reversibly over-stocked in capital

The case of a costly divestment is somewhat different as there is now the option to contract. For a given price b_n there are now two bench marks for stock levels. The first and lower value is the optimum level $\widehat{v}_n$ (computed as above) below which the stock level should not fall but there is now a second, larger, upper optimum level $\widehat{v}_n(\phi_n)$, dependent on the current resale rate, above which the stock should not rise.

At time t_n the optimal highest stock level worth keeping exists and is given by

$$\widehat{v}_n(\phi_n, b_n) = \frac{1}{(b_n\phi_n)^2} + \frac{1}{(b_n\phi_n g_n(\phi_n))^2}\left[1 + \frac{\gamma}{\widetilde{g}^2_{n+2,n+2}} + ... + \frac{\gamma^{N-n}}{\widetilde{g}^2_{n+2,N}}\right],$$

i.e. as though the current price was $\phi_n b_n$ and future prices were to be $b_{n+1} = \phi_n b_n \cdot g_n(\phi_n)$, $b_{n+2} = \phi_n b_n g_n(\phi_n)\widehat{g}_{n+2}, ..., b_m = \phi_n b_n \cdot g_n(\phi_n)\widetilde{g}_{n+2,m}, ...$. Corresponding to $\widehat{v}_n(\phi_n)$ there is an optimal current revenue from production, namely $1/(\phi_n b_n)$, and an optimal carry-forward $\widehat{u}_n(\phi_n, b_n) = \widehat{v}_n(\phi_n, b_n) - 1/(b_n\phi_n)$, i.e. of the form $\widehat{u}_n(\phi_n) = k/b_n^2$. As before the future value from carrying-forward is:

$$\overline{F}(\widehat{u}_n(\phi_n, b_n), b_n) = \frac{1}{b_n}\overline{F}(\widehat{u}_n(\phi_n, 1), 1)$$

which we can write as:

$$\widehat{V}^{\phi}_n = \frac{C^{\phi}_{n,N}}{b_n}.$$

This is linear in the production contribution $y_n = 1/(\phi_n b_n)$ to residual income. However, when price b_n is low enough so that the firm opening stock, v_n is above $\widehat{v}_n(\phi_n)$, there is an additional contribution to residual income arising from the optimal amount to sell off namely $[v_n - \widehat{v}_n(\phi_n)]$ whose value is:

$$b_n\phi_n[v_n - \widehat{u}_n(\phi_n) - \frac{1}{(b_n\phi_n)^2}].$$

The relationship between the full residual income Y_n given by

$$
\begin{aligned}
Y_n &= \frac{2}{\phi_n b_n} + b_n\phi_n[v_n - \widehat{u}_n(\phi_n) - \frac{1}{(b_n\phi_n)^2}] \\
&= \frac{1}{b_n}[\frac{1}{\phi_n} - k\phi_n] + b_n\phi_n v_n.
\end{aligned}
$$

Note that $Y_n \geq 2/(\phi_n b_n)$. Eliminating b_n as in the previous section, relates the future residual income to the expected future value of the project on the assumption that b_n is not empirically observable. We thus obtain

$$
Y_n = v_n\phi_n\frac{C^{\phi}_{n,N}}{V_n} - \frac{V_n}{C^{\phi}_{n,N}}[\frac{1}{\phi_n} - k\phi_n], \tag{29}
$$

which is thus hyperbolic, just as in the overstocked case, however for large b_n it is no longer the asymptotically linear arm of the hyperbola that is significant. Instead the dominant behaviour is an inverse linear relation:

$$
V_n \approx v_n\phi_n\frac{C^{\phi}_{n,N}}{Y_n},
$$

for $Y_n \geq 2/(\phi_n b_n)$.

Of course it is also the case that

$$
\begin{aligned}
TV &= V_n + Y_n \\
&= \frac{C^{\phi}_{n,N}}{b_n} + Y_n,
\end{aligned} \tag{30}
$$

which is a different linear relationship to (28) albeit a parallel line.

4.3 Regime (iii): Given $\phi < 1$ the firm is neither over invested nor under invested

In this intermediate input price range

$$
b_n(v, 1) < b_n < b_n(v, \phi_n), \tag{31}
$$

which corresponds to

$$
\frac{2}{b_n} < Y_n < \frac{2}{\phi_n b_n}
$$

the firm neither invests nor divests. It partitions its stock v_n into current optimal consumption $x_n(v_n, b_n)$ and investment carried forward $u_n(v_n, b_n)$. The residual income is thus

$$
Y_n = 2\sqrt{x_n(v_n, b_n)},
$$

a functional relation from which b_n can be deduced from the observable Y_n given knowledge of v_n.

Figure 2:

In the case when v_n is empirically observed and b_n is not, we can interpolate the graph of the future value of the firm, $V_n(u_n(v_n, b_n), b_n)$, against Y_n to fit between the two relations given implicitly by (27) and (29), and here the appropriate b_n range for the interpolation is given by (31). The resulting graph is given in Figure 2.

In the case where b_n is observed and v_n is not, the total value of the firm is

$$TV = V_n(u_n(v_n, b_n), b_n) + Y_n. \tag{32}$$

We show in the middle section of Figure 3 an example plot of TV against Y_n in the appropriate v_n range given by

$$\widehat{v}_n(b_n) < v_n < \widehat{v}_n(\phi_n, b_n).$$

That section of graph is in fact non-linear despite its appearance. Together the three relations (28), (30), (32) define an almost piecewise linear graph for the dependence of total value on residual income given in Figure 3.

The future optimal value of the firm as a function of residual income when b_n remains empirically unobserved is flat, joining the other two hyperbolic curves. To understand the nature of this functional form we invite the reader to consider an extreme case, where there is no divestment possible.

4.3.1 Regime (iiia): Special case of irreversibly overstocked project

We begin by assuming there is no option to contract and for simplicity we consider a three period model. Suppose the second period price is b_1 and the last period price is $b_2 = b_1 h$. The framework adopted naturally allows the project in the middle period to be overstocked. In such a case the current residual income is $f(x_1^*)$ since the optimal input given current opening stock v_1 is $x_1^* = x_1^*(v_1, b_1)$, is taken entirely from existing stock. This leaves $u_1^* = u_1^*(v_1, b_1)$ for the final period. As before, the

Figure 3:

future value is $\overline{F}(u_1^*, b_1)$ and as before

$$\overline{F}(u_1^*, b_1) = \frac{1}{b_1}\overline{F}(u_1^*(b_1)b_1^2, 1)$$

but its explicit dependence on b_1 is most easily traced through numeric work. Figure 2 graphs the relationship between $f(x_1^*)$ and the future value. It is clear what is going on; irreversibility in the final period leaves final residual income constant thus leading to a flattening of the curve. This feature is similar to the Kenton Yee (2000) graph. In order to develop a relatively simple generalisable intuition for the result, we develop the following informal argument. First we proxy the residual income of the second period $f(x_1^*)$ by $y_1 = b_1^{-1}$ (which thus equals $b_2^{-1}h$). This being a quantity of the right order of magnitude close to the frontier of this regime. Secondly, we assume some judiciously chosen fixed positive proportion of the stock is allocated to the final period (selected according to some expectation of the next period's price). As $b_1^{-1} \to 0$ it is the case that also $E[b_2^{-1}|b_1] \to 0$ and so in the limit the partitioning of the opening stock v_1 into current use $v_1 - u_1$ and final period use u_1 amounts to maximisation of the revenue

$$2[\sqrt{v_1 - u_1} + \gamma\sqrt{u_1}],$$

which leads to the first order condition

$$\frac{1}{\sqrt{v_1 - u_1}} = \frac{\gamma}{\sqrt{u_1}}$$

and so the partition of stock between current and future use is asymptotic to $1/\gamma^2 = (1+r)^2$. In any case the ratio of current to future use is between the asymptotic value and $\widehat{g}_1^2$.

So if the stock carried over to the final period is $u = u_1$ then we have understocking in the final period if $u < b_2^{-2}$, i.e. $y_1 = b_1^{-1} > h\sqrt{u}$; in this case the optimal profit is obtained by buying in extra stock where upon the final period profit is:

$$2\sqrt{b_2^{-2}} - b_2(b_2^{-2} - u) = b_2^{-1} + b_2 u = h^{-1}y_1 + huy_1^{-1}.$$

Now the minimum value of $b_2^{-1} + b_2 u$ occurs at $b_2 = 1/\sqrt{u}$ and is equal to $2\sqrt{u} = f(u)$. On the other hand, when $y_1 = b_1^{-1} < h\sqrt{u}$ the firm is overstocked and, in the absence of resale possibilities, all of the input is committed to production so that the revenue is constantly $f(u)$. Thus, given h the dependence of profit in the last period on previous period proxy residual income y_1 is given by:

$$\Pi_2(y_1|h) = \begin{cases} 2\sqrt{u} & \text{if } y_1 \leq h\sqrt{u}, \\ h^{-1}y_1 + huy_1^{-1} & \text{if } y_1 > h\sqrt{u}. \end{cases}$$

The graph of profit Π_2 against y_1 (for h fixed, and u fixed) is convex and initially flat and thereafter asymptotically linear. It follows immediately that $V(y_1) = E_h[\Pi_2(y_1|h)]$ is likewise convex (being a convex combination of convex functions) and for a log-normal distribution of h flat at the origin.

In the multiperiod case one argues similarly that as the price $b_n \to \infty$ so that $y_n = b_n^{-1} \to 0$ the limiting problem is to maximize:

$$V = 2(\sqrt{x_0} + \gamma\sqrt{x_1} + ... + \gamma^{n-1}\sqrt{x_{n-1}})$$

subject to

$$x_0 + x_1 + ... + x_{n-1} = v.$$

A simple Lagrangian analysis shows that again the future value V is proportional to $2\sqrt{v}$ and so the earlier observation of flatness near the origin does generalize to a multiperiod setting. In either case the more the project becomes over-invested the more the relationship flattens out.

Comment. In the general analysis for a Cobb-Douglas production function $f(x)$, the functions $x_n^*(v,b)$ and $u_n^*(v,b)$ can be derived from certain 'special functions' $\tilde{u}_n(v)$, which identify the optimal carry-forward when the current price is unity (so that the actual carry-forward $u_n(v,b_n)$ for a price b_n is obtainable by the substitution $u_n(v,b_n) = \tilde{u}_n(vb_n^{1/\alpha})b_n^{-1/\alpha}$ if the Cobb-Douglas index is α).

To summarise the above analysis, we see that, in contrast to the FO model, our real options model predicts the relationship between accounting residual profit and firm value will be approximately linear and non decreasing only if the firm is exercising the option to expand[26]. If it is exercising the option to contract or chooses not to exercise either option the FO model and our model is at odds concerning the link between firm market value and accounting residual income.

At this point it is interesting to note that a number of authors, as for instance reviewed in Yee (2000), have found empirically that for low accounting profit the non decreasing linear relationship in fact becomes convex which agrees with our model prediction. However in the model of Yee (2000) the link between residual income and firm value being convex arises for somewhat different reasons. In particular although his model is also a real options model, in his setting management operate within

[26] Recall the FO model implicitly assumes that the firm expands at a constant rate every period.

a FO framework but are subjected to project payoff shocks which means certain projects may earn only a small return. In such a case Yee argues that managers may switch out of poorly performing projetcs and hence the concavity arises precisely because managers discretely switch out of poorly performing projects. Thus the principal difference with our model is that he does not allow for the option to contract as we do; in his model managers must switch to some exogenously provided project if they are to not locate in the low valuation range whereas in our model setting they endogenously cut back investment in existing projects.

5 Conclusion

For the FO model recall that FO superimpose (4) and (5) on (3). However, as has been argued extensively above, superimposing this simple $AR(1)$ process on the way residual income grows, considerably restricts the type of underlying investment behaviour that could be consistent with the model. As Figure 2 illustrates, the objective of the previous section was to establish a more flexible model which will allow an alternative representation of $E_t(\widetilde{v}_{t+\tau})$ based upon optimal managerial real options evaluation. These findings are significant because the Feltham Ohlson valuation framework has been used by empiricists to test the value relevance of accounting data. Some researchers have criticised how empiricists have used the model to try to specify appropriate empirical testing procedures for the value relevance of accounting information. In contrast, we show how, independently of specifications issues, the underlying constant growth assumption which is central to the Feltham Ohlson framework removes the possibility for management to have a role in deciding whether or not to exercise expansion and contraction possibilities which do occur with most investment projects. Given this limitation we develop an alternative valuation framework which does not suffer from these limitations because the option to expand or contract optimally is given centre stage in our model of managerial decision making. We hope in a future paper to present a further refinement of our model in applied empirical settings which will aid researchers to re-appraise the value relevance of accounting data.

References

Abel, A.B., Dixit, A.K., Eberley, J.C. and Pindyck, R.S., (1996), “Options, the Value of Capital, and Investment”, Quarterley Journal of Economics.

Dixit, A.K., and Pindyck, R.S., (199*), “Decision Making Under Uncertainty”,

Gietzmann M.B., Ostaszewski A.J. (1999), Hedging the Purchase of Direct Inputs in an Inflationary Environment, Management Accounting Research, to appear.

Gietzmann M.B., Ostaszewski A.J. (1999), The N-fold Replenishment Problem, LSE-CDAM Research Reports, in preparation.

Gietzmann M.B., Ostaszewski A.J. (2000), Endogenous Irreversibility With Finite Horizon Investment When resale is Possible, LSE-CDAM Research Reports, 2000-16.

K. Lo and T. Lys (2000), “The Ohlson Model: Contribution to Valuation Theory, Limitations and Empirical Applications, *Journal of Accounting Auditing and Finance.*

L. Trigeorgis (1996), “Real Options”, MIT Press, Cambridge.

K.K. Yee (2000), “Opportunities Knocking: Residual Income Valuation of an Adaptive Firm”, *Journal of Accounting Auditing and Finance*.

M. Walker, (1997), “Clean Surplus Accounting Models and Market Based Accounting Research: A Review”, *Accounting and Business Research*.

6 Appendix A: NPV Rule

In this section we derive the NPV Rule. This follows from results obtained in from Gietzmann M.B., Ostaszewski A.J. (2000) page 42. The notation is as follows: $F(v, b_0, \phi)$ denotes the discounted future maximum expected profit ignoring the historic cost of the carry–forward input v. Thus, for example $F_0(v, b_0, \phi) = \gamma V_0(v, b_0, \phi)$ and the corresponding value of the firm ignoring past costs and revenues is $V_0(v|b_0) = f(G(b_0)) - b_0 G(b_0) + F_1(v, b_0, \phi)$. Now we have

$$\begin{aligned}
\gamma^{-1} F(v, b_0, \phi) &= \int_0^{b(v,1)} \left[f^{\#}(G(b_1)) + b_1(v - \widehat{u}(1, b_1)) + \overline{F_+}(\widehat{u}(1, b_1), b_1) \right] dQ_1 \\
&\quad + \int_{b(v,1)}^{b(v,\phi)} \left[f(v - u(v, b_1)) + \overline{F_+}(u(v, b_1), b_1) \right] dQ_1 \\
&\quad + \int_{b(v,\phi)}^{\infty} \left[\phi b_1(v - \widehat{u}(\phi, b_1)) + f^{\#}(G(\phi b_1)) + \overline{F_+}(\widehat{u}(\phi, b_1), b_1) \right] dQ_1.
\end{aligned}$$

Hence proceeding formally and applying the Liebniz Rule[27]

$$\begin{aligned}
\gamma^{-1} F'(v, b_0, \phi) &= \int_0^{b(v,1)} b_1 dQ_1 \\
&\quad + \int_{b(v,1)}^{b(v,\phi)} \left[f'(v - u(v, b_1))(1 - u') + \overline{F_+}'(u(v, b_1), b_1) u' \right] dQ_1 \\
&\quad + \int_{b(v,\phi)}^{\infty} \phi b_1 dQ_1.
\end{aligned}$$

But $f'(v - u(v, b_1)) = \overline{F_+}'(u(v, b_1), b_1)$ by definition of $u(v, b_1)$. So

$$F'(v, b_0, \phi) = \int_0^{b(v,1)} b_1 dQ_1 + \int_{b(v,1)}^{b(v,\phi)} f'(v - u(v, b_1)) dQ_1 + \int_{b(v,\phi)}^{\infty} \phi b_1 dQ_1.$$

Hence

$$\begin{aligned}
\gamma^{-1} F'(u, b_0, \phi) &= \int_0^{b(u,1)} b_1 dQ_1 + \int_{b(u,1)}^{b(u,\phi)} f'(x(u, b_1)) dQ_1 + \int_{b(u,\phi)}^{\infty} \phi b_1 dQ_1, \\
&= \int_0^{\widetilde{b}_1} b_1 dQ_1 + \int_{\widetilde{b}_1}^{h_1(\widetilde{b}_1,\phi)} f'(x(u, b_1)) dQ_1 + \int_{h_1(\widetilde{b}_1,\phi)}^{\infty} \phi b_1 dQ_1, \\
&= \widetilde{b}_1 + \int_0^{\widetilde{b}_1} (b_1 - \widetilde{b}_1) dQ_1 + \int_{\widetilde{b}_1}^{h_1(\widetilde{b}_1,\phi)} (f'(x(u, b_1)) - \widetilde{b}_1) dQ_1 \\
&\quad + \int_{h_1(\widetilde{b}_1,\phi)}^{\infty} (\phi b_1 - \widetilde{b}_1) dQ_1.
\end{aligned}$$

[27] We do not show cancelling terms.

7 Appendix B: The optimal replenishment policy.

We prove the Recurrence Lemma

$$\hat{u}_{n+1}(b_n, \phi_{n+1}) = G(\hat{b}_{n+1}(b_n)) + \hat{u}_n(\hat{b}_{n+1}(b_n), \phi_{n+2}).$$

Proof. Recall that the function $u = \hat{u}_n(b, \phi)$ is defined by the equation[28]

$$F'_n(u, b) = \phi b.$$

We work inductively. To obtain the solution $v = \hat{u}_{n+1}(b_n, \phi_{n+1})$of the first order condition

$$F'_{n+1}(v, b_n) = \phi_{n+1} b_n.$$

we begin by first solving the censor equation

$$q_n(B, \phi_{n+1}, , b_n) = \phi b_n.$$

We denote the solution[29] by $\hat{b}_{n+1}(b_n, \phi)$. Recall that

$$\begin{aligned} q_m(B, \phi_{m+1}, b_m) &= \int_0^B b_m dQ(b_m|b_{m-1}) \\ &+ \int_B^{h_m(B,\phi_{m+1})} f'(x_m(v_B^m, b_m))dQ(b_m|b_{m-1}) + \phi_{m+1} \int_{h_m(B,\phi_{m+1})}^{\infty} b_m dQ(b_m|b_{m-1}), \end{aligned}$$

Here $v_b^m = G(b) + \hat{u}_m(b, \phi_{m+1})$, and $G(b) = \{f'\}^{-1}(b)$. Now for an appropriate function $B_{n+1}(v)$ we have

$$F'_{n+1}(v, b_n) = q_n(B_{n+1}(v), b_n),$$

so we now need to solve

$$B_{n+1}(\widehat{v}) = \hat{b}_{n+1}(b_n, \phi).$$

But recalling that in general $F(v, b) = f(v - u(v, b) + F_+(u, b)$ we have

$$\begin{aligned} F'_{n+1}(v_b, b) &= f'(v_b - u(v_b^{n+1}, b))(1 - u') + F'_{n+2}(u, b)u' \\ &= f'(v_b - u(v_b^{n+1}, b)) = b. \end{aligned}$$

Thus we have the identity

$$B_{n+1}(v_B^{n+1}) = F'_n(v_B^{n+1}, B_{n+1}(v_B^{n+1})) = B.$$

[28] Recall the convention that $F = \gamma V$.

[29] In the Cobb-Douglas case

$$\hat{b}_{n+1}(b_n) = \phi b_n \widehat{g}_n(\phi)$$

for some constant $\widehat{g}_n(\phi)$.

Hence for $B = \hat{b}_{n+1}(b_n, \phi)$ we have identified that $\widehat{v} = v_B^{n+1}$. In conclusion we have

$$\begin{aligned} \hat{u}_{n+1}(b_n, \phi) &= G(\hat{b}_{n+1}(b_n, \phi)) + \hat{u}_n(\hat{b}_{n+1}(b_n, \phi), 1) \\ &= G(\hat{b}_{n+1}(b_n, \phi)) + G(\hat{b}_{n+2}(\hat{b}_{n+1}(b_n, \phi), 1)) + ... \end{aligned}$$

Corollary. The analysis prescribes an aggregate demand of

$$D_n^{\phi}(b_n) = G(\hat{b}_{n+1}) + ... + G(\hat{b}_i) + ...,$$

where $G(b) = \{f'\}^{-1}(b)$ and the sequence $\hat{b}_{n+i}$ is given by the iteration

$$\begin{aligned} \hat{b}_{n+1} &= \hat{b}_{n+1}(b_n, \phi), \\ \hat{b}_{n+2} &= \hat{b}_{n+1}(\hat{b}_{n+1}, 1), \\ \hat{b}_{n+3} &= \hat{b}_{n+3}(\hat{b}_{n+2}, 1), \\ & \quad ... \end{aligned}$$

It is now easy to describe the replenishment programme. Suppose we have n periods remaining and we have a stock v. The acquisition programme calls for an optimal aggregate demand to be purchased of

$$D_n^1(b_n) = G(\hat{b}_{n+1}) + ... + G(\hat{b}_i) + ...,$$

(i.e. with $\phi = 1$) and either we have v below this amount in which case we need to top up to this amount or else we are moderately over-stocked and must carry-forward $u_n^*(v, \phi_{n+1}, b_n)$ without selling, or else we must sell to the point where the stock is $D_n^{\phi_{n+1}}(b_n)$.Thus

$$u_n^*(v, b_n) = \begin{cases} \widehat{u}_n(1, b_n) & b_n < b_n(v, 1) \\ u(v, b_n) & b_n(v, 1) < b_n < b_n(v, \phi_{n+1}) \\ \widehat{u}_n(\phi, b_n) & b_n(v, \phi_{n+1}) < b_n \end{cases}$$

8 Appendix C: Derivation of Valuation formula

We study first the general two stage situation. The current price is b_0 the next period price is b_1 and the resale rate is ϕ. Our notation in this section for the maximum expected value given a stock v of inputs and given the knowledge of $\phi = \phi_1$ is $F(v, b_0, \phi)$; once b_1 is revealed and u is carried forward into the future the maximum expected revenue from the period beyond is $\overline{F_+}(u, b_1)$ where the bar signifies expectation over ϕ_2.

We note that $F = \gamma V$.

8.1 Step 1. We prove a recurrence

$$\gamma^{-1} F(v, b_0, \phi) = E_{b_1}[f^{\#}(x^*(v, b_1)) + \overline{F_+}^{\#}(u^*(v, b_1), b_1)] + \gamma v q$$

where the notation is as in section 3.2 above and is recalled below.

Proof. We have (from Gietzmann M.B., Ostaszewski A.J. (2000) page 42) that

$$\begin{aligned}\gamma^{-1}F(v,b_0,\phi) &= \int_0^{b(v,1)} \left[f(G(b_1)) - b_1G(b_1) + b_1(v-\widehat{u}(1,b_1)) + \overline{F_+}(\widehat{u}(1,b_1),b_1)\right] dQ_1 \\ &+ \int_{b(v,1)}^{b(v,\phi)} \left[f(v-u(v,b_1)) + \overline{F_+}(u(v,b_1),b_1)\right] dQ_1 \\ &+ \int_{b(v,\phi)}^{\infty} \left[\phi b_1(v-\widehat{u}(\phi,b_1)) + f^{\#}(G(\phi b_1)) + \overline{F_+}(\widehat{u}(\phi,b_1),b_1)\right] dQ_1.\end{aligned}$$

To understand the first integral (corresponding to the understocked situation), note that the additional purchase z is specified by $v+z = G(b_1)+\widehat{u}(1,b_1)$ and so the revenue is $f(G(b_1)) - b_1(G(b_1) + \widehat{u}(1,b_1) - v)$.

Now we reorganize the expression on the right. First note that $f^{\#}(x) = f(x) - xf'(x)$ and since G is the inverse of f' we have

$$f^{\#}(G(b_1)) = f(G(b_1)) - b_1G(b_1).$$

Similarly $\overline{F_+}^*(x) = \overline{F_+}(x) - x\overline{F_+}'(x)$. But since $u = \widehat{u}(1,b_1)$ solves

$$b_1 = \gamma V_+'(u,b_1) = \overline{F_+}'(u,b_1)$$

we have

$$\overline{F_+}^{\#}(\widehat{u}(1,b_1),b_1) = \overline{F_+}(\widehat{u}(1,b_1),b_1) - b_1\widehat{u}(1,b_1).$$

Likewise

$$\overline{F_+}^{\#}(\widehat{u}(\phi,b_1),b_1) = \overline{F_+}(\widehat{u}(\phi,b_1),b_1) - \phi b_1\widehat{u}(\phi,b_1).$$

Lastly $u = u(v,b_1)$ solves

$$f'(v-u) = \gamma\overline{V_+}'(u,b_1) = \overline{F_+}'(u,b_1)$$

hence

$$\begin{aligned}\overline{F_+}^{\#}(u(v,b_1),b_1) &= \overline{F_+}(u(v,b_1),b_1) - u(v,b_1)f'(v-u(v,b_1)) \\ &= \overline{F_+}(u(v,b_1),b_1) - u(v,b_1)f'(x(v,b_1))\end{aligned}$$

where $x(v,b_1) = v - u(v,b_1)$. Of course

$$f^{\#}(x(v,b_1)) = f(x(v,b_1)) - x(v,b_1)f'(x(v,b_1))$$

We thus have, writing x for $x(v,b_1)$

$$\begin{aligned}\gamma^{-1}F(v,b_0,\phi) &= \int_0^{b(v,1)} \left[f^{\#}(G(b_1)) + \overline{F_+}^{\#}(\widehat{u}(1,b_1),b_1)\right] dQ_1 + v\left[\int_0^{b(v,1)} b_1 dQ_1\right] \\ &+ \int_{b(v,1)}^{b(v,\phi)} \left[f^{\#}(x) + \overline{F_+}^{\#}(u(v,b_1),b_1)\right] dQ_1 + \int_{b(v,1)}^{b(v,\phi)} [xf'(x) + u(v,b_1)f'(x)]\, dQ_1 \\ &+ \int_{b(v,\phi)}^{\infty} \left[f^{\#}(G(\phi b_1)) + \overline{F_+}^{\#}(\widehat{u}(\phi,b_1),b_1)\right] dQ_1 + v\left[\int_{b(v,\phi)}^{\infty} \phi b_1 dQ_1\right]\end{aligned}$$

or just

$$
\begin{aligned}
\gamma^{-1}F(v,b_0,\phi) &= \int_0^{b(v,1)} \left[f^{\#}(G(b_1)) + \overline{F_+}^{\#}(\widehat{u}(1,b_1),b_1)\right] dQ_1 \\
&\quad + \int_{b(v,1)}^{b(v,\phi)} \left[f^{\#}(x) + \overline{F_+}^{\#}(u(v,b_1),b_1)\right] dQ_1 \\
&\quad + \int_{b(v,\phi)}^{\infty} \left[f^{\#}(G(\phi b_1)) + \overline{F_+}^{\#}(\widehat{u}(\phi,b_1),b_1)\right] dQ_1 \\
&\quad + v\left[\int_0^{b(v,1)} b_1 dQ_1 + \int_{b(v,1)}^{b(v,\phi)} f'(x)dQ_1 + \int_{b(v,\phi)}^{\infty} \phi b_1 dQ_1\right]
\end{aligned}
$$

and this may be rendered in a more compact way as asserted above, namely :

$$
\gamma^{-1}F(v,b_0,\phi) = E_{b_1}[f^{\#}(x^*(v,b_1)) + \overline{F_+}^{\#}(u^*(v,b_1),b_1)] + vq
$$

provided we introduce the notation

$$
x^*(v,b_1) = \begin{cases} G(b_1) & b_1 < b(v,1) \\ x(v,b_1) & b(v,1) < b_1 < b(v,\phi) \\ G(\phi b_1) & b(v,\phi) < b_1 \end{cases}
$$

and

$$
u^*(v,b_1) = \begin{cases} \widehat{u}(1,b_1) & b_1 < b_1(v,1) \\ u(v,b_1) & b_1(v,1) < b_1 < b_1(v,\phi) \\ \widehat{u}(\phi,b_1) & b_1(v,\phi) < b_1 \end{cases}
$$

where

$$
x(v,b_1) = v - u(v,b_1)
$$

and

$$
q = \int_0^{b_1(v,1)} b_1 dQ_1 + \int_{b_1(v,1)}^{b_1(v,\phi)} f'(x(v,b_1))dQ_1 + \int_{b_1(v,\phi)}^{\infty} \phi b_1 dQ_1.
$$

It is convenient to define a function h_1 by the simultaneous equations

$$
\left.\begin{aligned} h_1(B,\phi) = b_1(v,\phi), \\ B = b_1(v,1). \end{aligned}\right\}
$$

i.e. $h_1(B,\phi) = b_1(v_B^1,\phi)$ where $v = v_B^1$ solves $B = b_1(v,1)$. We identify these functions in the Cobb-Douglas case in a later section. In conclusion we may define an important function $q_0(B)$ as follows.

$$
q_0(B) = \int_0^{B} b_1 dQ_1 + \int_{B}^{h_1(B,\phi)} f'(x(v_B,b_1))dQ_1 + \int_{h_1(B,\phi)}^{\infty} \phi b_1 dQ_1.
$$

The solution for B of $q_0(B) = b_0$ is the censor $\widetilde{b}_1 = \widetilde{b}_1(b_0)$.

8.2 Step 2. Deduction from the Recurrence

We prove

$$V_0(v,b_0,\phi) = E[\sum_{n=1}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))] + vq_0(b_1(v,1)).$$

or

$$F_0(v,b_0,\phi) = E[\sum_{n=1}^{N} \gamma^{n} f^{\#}(x_n^*(b_n))] + \gamma vq_0(b_1(v,1)).$$

Proof. We already know that

$$\gamma^{-1}F'(v,b_0) = V' = q(b(v,1)) = \int_0^{b(v,1)} bdQ(b) + \int_{b(v,1)}^{b(v,\phi)} f'(x(v,b))dQ(b) + \int_{b(v,\phi)}^{\infty} \phi bdQ(b)$$

(using generic notation), hence we may also write

$$\begin{aligned} V - vq &= \gamma^{-1}[F - vq\gamma] \\ &= \gamma^{-1}[F(v,b_0,\phi) - vF'(v,b_0)] \\ &= \gamma^{-1}F^{\#}(v,b_0,\phi) \\ &= E_{b_1}[f^{\#}(x^*(v,b_1)) + \overline{F_+}^*(u^*(v,b_1),b_1)] \end{aligned}$$

and taking expectations over ϕ we have

$$\gamma^{-1}\overline{F}^*(v,b_0) = E_{\phi,b_1}[f^{\#}(x^*(v,b_1)) + \overline{F_+}^*(u^*(v,b_1),b_1)].$$

We may now apply this result inductively, the first steps being

$$\begin{aligned} \gamma^{-1}F_0^{\#}(v,b_0,\phi) &= E_{b_1}[f^{\#}(x_1^*(v,b_1)) + \overline{F_1}^{\#}(u_1^*(v,b_1),b_1)] \\ &= E_{b_1}[f^{\#}(x_1^*(v,b_1)) + \gamma E_{b_2}[f^{\#}(x_2^*(u_1^*,b_2)) + \overline{F_2}^{\#}(u_2^*(u_1^*,b_2),b_2)]] \\ &= E_{b_1}[f^{\#}(x_1^*(v,b_1)) + \gamma E_{b_2}[f^{\#}(x_2^*(u_1^*,b_2)) + \gamma E_{b_3}[f^{\#}(x_3^*(u_2^*,b_3)) + \overline{F_3}^{\#}(u_3^*(u_2^*,b_3),b_2)]]] \\ &= E_{b_1}[f^{\#}(x_1^*(v,b_1)) + E_{b_2}[\gamma f^{\#}(x_2^*(u_1^*,b_2)) + E_{b_3}[\gamma^2 f^{\#}(x_3^*(u_2^*,b_3)) + \gamma^2\overline{F_3}^{\#}(u_3^*(u_2^*,b_3),b_2)]]] \end{aligned}$$

Assuming N steps so that $F_{N+1} = 0$ we obtain on suppressing some notation that

$$\gamma^{-1}F_0^{\#}(v,b_0,\phi_1) = E[\sum_{n=1}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))].$$

so

$$V_0(v,b_0,\phi_1) - vq_1 = E[\sum_{n=1}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))].$$

Rewriting we obtain the valuation

$$\begin{aligned}
& f^{\#}(x_0^*(v,b_0)) + \gamma V_0(v,b_0,\phi_1) \\
= \quad & f^{\#}(x_0^*(v,b_0)) + E[\sum_{n=1}^{N} \gamma^n f^{\#}(x_n^*(b_n))] + v\gamma q_1(b_1(v,1)),
\end{aligned}$$

where

$$q_0(B) = \int_0^B b_1 dQ_1 + \int_B^{h_1(B,\phi)} f'(x(v_B^1, b_1))dQ_1 + \int_{h_1(B,\phi)}^{\infty} \phi b_1 dQ_1.$$

In the next section we identify the functional form for the Cobb-Douglas case.

9 Appendix D: The Cobb-Douglas case

In this section we give the explicit form of all relevant auxiliary functions needed to compute the terms in the accounting identity. In particular we recall relevant formulas and results derived in our CDAM paper apropriate to the case $f(x) = x^{1-\alpha}/(1-\alpha)$ when $0 < \alpha < 1$. Note that $f' = x^{-\alpha}$ has inverse $G(b) = b^{-1/\alpha}$ and so

$$\begin{aligned}
f^{\#}(x) &= \frac{\alpha}{1-\alpha} x^{1-\alpha}, \\
f^{\#}(G(b)) &= \frac{\alpha}{1-\alpha} b^{-(1-\alpha)/\alpha}.
\end{aligned}$$

The homogeneity property is given by:

$$\overline{V}'_n(u, b_n) = b_n \overline{V}'_n(u b_n^{1/\alpha}, 1),$$

and more interestingly by

$$\overline{V}'_n(u, b_n) = B \overline{V}'_n(u B^{1/\alpha}, b_n/B).$$

Thus for $\phi = \phi_n$ or $\phi = 1$ the solution $u = \hat{u}(b_n, \phi)$ to $\overline{V}'_n(u) = \overline{V}'_n(u, b_n) = b_n \phi$ is of the form $u b_n^{1/\alpha} = \overline{u}$ (i.e. $u = \overline{u} b_n^{-1/\alpha}$) where $\overline{u} = \overline{u}_n(\phi)$ solves

$$\overline{V}'_n(\overline{u}_n, 1) = \phi,$$

assuming

$$\phi > \inf_u \overline{V}'_n(u, 1),$$

otherwise there is no solution (and therefore no need to sell-back stock).

It may be shown quite generally (see Appendix B) that

$$\begin{aligned}
\hat{u}_{n+1}(b_n, \phi) &= G(\hat{b}_{n+1}(b_n, \phi)) + \hat{u}_{n+2}(\hat{b}_{n+1}(b_n, \phi), 1) \\
&= (\hat{b}_{n+1}(b_n))^{-1/\alpha} + \hat{u}_{n+2}(\hat{b}_{n+1}(b_n), 1)
\end{aligned}$$

just as in (24).

Now the equation

$$\overline{V}'_n(\overline{u}_n, 1) = \phi,$$

is equivalent (see Appendix B) under a transformation of variables to

$$q_n(B, b_n) = \phi b_n$$

and trhis in turn to

$$q_n(B/b_n, 1) = \phi.$$

So we let $g = g_{n+1}(\phi)$ solve

$$q_n(g\phi, 1) = \phi,$$

with the convention[30] that $g_{n+1}(\phi) = 0$ when

$$\phi < \inf_g q_n(g, 1).$$

Thus the original equation for B is solved by setting

$$B/b_n = g_{n+1}(\phi)\phi$$

i.e. $B = g_n(\phi)(\phi b_n)$. We thus have

$$\begin{aligned} \hat{u}_{n+1}(b_n, \phi) &= (g_{n+1}(\phi)\phi b_n)^{-1/\alpha} + (g_{n+2}(1)g_{n+1}(\phi)\phi b_n)^{-1/\alpha} \\ &\quad + (g_{n+3}(1)g_{n+2}(1)g_{n+1}(\phi)\phi b_n)^{-1/\alpha} \end{aligned}$$

and

$$\begin{aligned} \widehat{v}_n(b_n, \phi) &= (\phi b_n)^{-1/\alpha} + (g_{n+1}(\phi)\phi b_n)^{-1/\alpha} + (g_{n+2}(1)g_{n+1}(\phi)\phi b_n)^{-1/\alpha} + ... \\ &= (\phi b_n)^{-1/\alpha}\left[1 + (g_{n+1}(\phi))^{-1/\alpha} + (g_{n+2}(1)g_{n+1}(\phi))^{-1/\alpha} + ...\right] \end{aligned} \quad (33)$$

By (33) we have

$$v^n_{\phi b} = (\kappa_n(\phi b))^{-1/\alpha}$$

where

$$\kappa_n(\phi)^{-1/\alpha} = 1 + (g_{n+1}(\phi))^{-1/\alpha} + (g_{n+2}(1)g_{n+1}(\phi))^{-1/\alpha} + ... \quad (34)$$

Note that $\kappa_N \equiv 1$.Thus the solution $b = b_n(v, \phi)$ to $v = v^n_{\phi b}$ is

$$b_n(v, \phi) = \phi^{-1}\kappa_n(\phi)^{-1}v^{-\alpha}.$$

Note the identity

$$b_{n+1}(\hat{u}_{n+1}(b_n, \phi), 1) = \phi g_{n+1}(\phi)b_n. \quad (35)$$

[30]This ensures that the bench-mark stock, above which all is to be sold is infinity, in keeping with the idea that there should be no resale.

This is evident if we notice that we have to solve

$$\begin{aligned}
\hat{u}_{n+1}(b_n, \phi) &= \widehat{v}_{n+1}(b_{n+1}, \phi) \\
&= (b_{n+1})^{-1/\alpha}\left[1 + (g_{n+2})^{-1/\alpha} + (g_{n+3}g_{n+2})^{-1/\alpha} + ...\right] \\
&= (g_{n+1}(\phi)\phi b_n)^{-1/\alpha} + (g_{n+2}(1)g_{n+1}(\phi)\phi b_n)^{-1/\alpha} + ...
\end{aligned}$$

If the project is overstocked, the carry-forward equation

$$f'(v-u) = \overline{V}'_n(u, b_n) \tag{36}$$

may be re-written using homogeneity as

$$f'(\widetilde{v} - \widetilde{u}(\widetilde{v})) = \overline{V}'_n(ub_n^{1/\alpha}, 1)$$

where $\widetilde{u} = ub_n^{1/\alpha}, \widetilde{v} = vb_n^{1/\alpha}$ or in standardised form as

$$f'(\widetilde{v} - \widetilde{u}(\widetilde{v})) = \overline{V}'_n(\widetilde{u}, 1)$$

with solution $\widetilde{u}_n(\widetilde{v})$. The solution of (36) is then $u_n(v, b_n) = \widetilde{u}(vb_n^{1/\alpha})b_n^{-1/\alpha}$. Note also

$$f'([v/u] - 1) = \overline{V}'_n(1, b_n u^\alpha)$$

so the utilization ratio

$$\frac{v}{u} = 1 + G(\overline{V}'_n(1, \lambda_n b_n/b_{n+1}(u)))$$

is a function of the ratio of the current price and the top-up limit. Here λ_n is a constant.

Evidently the special functions $\widetilde{u}_n(\widetilde{v})$ need numeric evaluation. They are defined inductively as follows. The base of the induction is:

$$\begin{aligned}
x_N(v, b_N) &= v, \\
u_{N+1}(v) &= 0, \\
q_{N-1}(B) &= \int_0^B b_N dQ(b_N) + \int_B^{B\psi_N} f'(x_N(v_B, b_N))dQ(b_N) + \phi_N \int_{B\psi_N}^\infty b_N dQ(b_N), \\
\psi_N &= 1/\phi_N, \\
v_B &= 1/B^2, \\
b_N(u, 1) &= 1/\sqrt{u}, \\
W_{N-1}(u) &= u + [q_{N-1}(b_N(u))]^{-2}, \\
\widetilde{u}_N(v) &= W_{N-1}^{-1}(v).
\end{aligned}$$

The inductive step is very similar.

$$
\begin{aligned}
x_n(v,b_n) &= v-\widetilde{u}_{n+1}(v),\\
q_{n-1}(B) &= \int_0^B b_n dQ(b_n)+\int_B^{B\psi_n} f'(x_n(v_B^n,b_n))dQ(b_n)+\phi_n\int_{B\psi_n}^\infty b_n dQ(b_n),\\
\psi_n &= \phi_n^{-1}\kappa_n(\phi_n)^{-1}\kappa_n(1)\\
v_B^n &= \frac{1}{\kappa_n(1)^2B^2},\\
b_n(v,1) &= \kappa_n(1)^{-1}v^{-1/2}\\
W_{n-1}(u) &= u+[q_{n-1}(b_n(u,1))]^{-2},\\
\widetilde{u}_n(v) &= W_{n-1}^{-1}(v).
\end{aligned}
$$

It is important to notice that the definition of κ_n calls for values known from earlier in the induction namely the numbers $g_m(\phi_m)$ for $m>n$. (See (34) above.)

However, before one can use these special functions, we need to know just when to apply them, i.e when and how much stock to resell. With this in mind, recall the definition of the functions h_m given by by the simultaneous equations

$$
\left.\begin{array}{c}
h_m(B,\phi)=b_m(v,\phi)=\phi^{-1}\kappa_m(\phi)^{-1}v^{-\alpha},\\
B=b_m(v,1)=\kappa_m(1)^{-1}v^{-\alpha}.
\end{array}\right\}
$$

Solving we obtain

$$
\begin{aligned}
h_m(B,\phi) &= \phi^{-1}\kappa_m(\phi)^{-1}\kappa_m(1)B=\psi_m B\\
&= h_m(1,\phi)B,
\end{aligned}
$$

so that, as asserted earlier in the finite horizon section, the dependence on B is linear. Note that $\psi_N=\phi_N^{-1}$.

As for the carry-forward we have the explicit forms:

$$
u^*_{n+1}(v,\phi_{n+1},b_n)=\begin{cases}
(\kappa_n(1)^{-1/\alpha}-1)b_n^{-1/\alpha} & b_nv^\alpha<\kappa_n(1)^{-1}\\
u_{n+1}(v,b_n) & \kappa_n(1)^{-1}<b_nv^\alpha<\phi_{n+1}{}^{-1}\kappa_n(\phi_{n+1})^{-1}\\
(\kappa_n(\phi_{n+1})^{-1/\alpha}-1)b_n^{-1/\alpha} & \phi_{n+1}^{-1}\kappa_n(\phi_{n+1})^{-1}<b_nv^\alpha
\end{cases}
$$

and

$$
x^*_n(v,\phi_{n+1},b_n)=\begin{cases}
b_n^{-1/\alpha} & b_nv^\alpha<\kappa_n(1)^{-1}\\
x_n(v,b_1) & \kappa_n(1)^{-1}<b_nv^\alpha<\phi_{n+1}{}^{-1}\kappa_n(\phi_{n+1})^{-1}\\
(\phi_{n+1}b_1)^{-1/\alpha} & \phi_{n+1}{}^{-1}\kappa_m(\phi_{n+1})^{-1}<b_nv^\alpha
\end{cases}
$$

10 Appendix E: Linear dependence of profits on output

In this appendix we prove that in the Cobb-Douglas case that

$$
\overline{F}(v\cdot G(b_0),b_0)=\overline{F}(v,1)f^{\#}(G(b_0)),
$$

so that in the square-root case we have[31]

$$\overline{F}(vb_0^{-2}, b_0) = \frac{1}{b_0}\overline{F}(v,1).$$

Our main conclusion is the result that:

$$\overline{F}(\widehat{v}(b_0), b_0) = \frac{1}{b_0}\overline{F}(\widehat{v}(1),1)$$

which asserts that for an optimally carried forward stock, the future expected indirect profits are linearly dependent on current indirect profit.

We note that in general, if $f(x) = x^{1-\alpha}/(1-\alpha)$, so that $f^{\#}(G(b)) = \frac{\alpha}{1-\alpha}b^{-(1-\alpha)/\alpha}$, then the formula at the head of this section in explicit terms is as follows.

$$\overline{F}(vb_0^{-1/\alpha}, b_0) = \frac{\alpha}{1-\alpha}b_0^{-(1-\alpha)/\alpha}\overline{F}(v,1).$$

For notation see section ?? above.

Proof. For transparency we write the proof in the square-root case.

We again refer to the formula (from Gietzmann M.B., Ostaszewski A.J. (2000) page 42):

$$\begin{aligned}
\gamma^{-1}F(v,b_0,\phi) &= \int_0^{b(v,1)} \left[f^{\#}(G(b_1)) + b_1(v-\widehat{u}(1,b_1)) + \overline{F_+}(\widehat{u}(1,b_1),b_1)\right] dQ_1 \\
&\quad + \int_{b(v,1)}^{b(v,\phi)} \left[f(v-u(v,b_1)) + \overline{F_+}(u(v,b_1),b_1)\right] dQ_1 \\
&\quad + \int_{b(v,\phi)}^{\infty} \left[\phi b_1(v-\widehat{u}(v,b_1)) + f^{\#}(G(\phi b_1)) + \overline{F_+}(\widehat{u}(\phi,b_1),b_1)\right] dQ_1.
\end{aligned}$$

We begin by assuming inductively the property that for all $v > 0$

$$\overline{F_+}(vg^{-2}b_1^{-2}, gb_1) = \frac{1}{b_1}\overline{F_+}(vg^{-2}, g)$$

and show that for all v we have

$$\overline{F}(vg^{-2}b_0^{-2}, gb_0) = \frac{1}{b_0}\overline{F}(vg^{-2}, g).$$

In the formula above replace b_0 by $b_0 g$ and v by $vg^{-2}b_0^{-2}$. We also make the substitution $h = b_1/(gb_0)$. We now factorize out b_0^{-1} using inductive assumptions and some simple manipulations. To see this done note the following calculations. First note that since $b(v,1) = K/\sqrt{v}$ (for some constant K) we have $b(v(gb_0)^{-2},1) = Kb_0/\sqrt{vg^{-2}} = b(vg^{-2},1)b_0$. Next we have

$$\begin{aligned}
& \overline{F_+}(u(vg^{-2}b_0^{-2}, b_1), b_1) \\
= \; & \overline{F_+}(u(vg^{-2}b_0^{-2}, hgb_0), hgb_0) \\
= \; & \overline{F_+}(\tilde{u}(vg^{-2}b_0^{-2}(hgb_0)^2)(hgb_0)^{-2}, hgb_0) \\
= \; & \overline{F_+}(\tilde{u}(vg^{-2}(hg)^2)(hgb_0)^{-2}, hgb_0) \\
= \; & \overline{F_+}(\tilde{u}(vg^{-2}(hg)^2)(hg)^{-2}, hg)/b_0 \\
= \; & \overline{F_+}(u(vg^{-2}, hg), hg)/b_0
\end{aligned}$$

[31] Thus $H(w,b) = F(1/w^2, b)$ is homogeneous of degree -1.

Similarly

$$\begin{aligned}
& f(vg^{-2}b_0^{-2} - u(vg^{-2}b_0^{-2}, hgb_0)) \\
= \;& f(vg^{-2}b_0^{-2} - \widetilde{u}(vg^{-2}b_0^{-2}h^2g^2b_0^2)h^{-2}g^{-2}b_0^{-2}) \\
= \;& b_0^{-1} f(vg^{-2} - \widetilde{u}(vg^{-2}h^2g^2)h^{-2}g^{-2}) \\
= \;& b_0^{-1} f(vg^{-2} - u(vg^{-2}, hg)).
\end{aligned}$$

(Since $u_n(v, b_n) = \widetilde{u}(vb_n^{1/\alpha})b_n^{-1/\alpha}$). Finally

$$\begin{aligned}
& \overline{F_+}(\widehat{u}(1, gb_0h), gb_0h) \\
= \;& \overline{F_+}(\widehat{u}(1,1)(gb_0h)^{-2}, gb_0h) \\
= \;& \overline{F_+}(\widehat{u}(1,1)(gh)^{-2}, gh)b_0^{-1} \\
= \;& \overline{F_+}(\widehat{u}(1, gh), gh)b_0^{-1}.
\end{aligned}$$

We thus obtain (dropping the display of the third term in view of its similarity to the first) that

$$\begin{aligned}
& \gamma^{-1}F(vg^{-2}b_0^{-2}, gb_0, \phi) \\
= \;& \int_0^{b(vg^{-2},1)} \left[\frac{1}{gb_0h} + gb_0h\left(\frac{1}{(gb_0h)^2} - \frac{1}{g^2b_0^2}\widehat{u}(1, gh)\right) + \overline{F_+}(\widehat{u}(1, gb_0h), gb_0h) \right] dQ_1(h) \\
& + \int_{b(vg^{-2},1)}^{b(vg^{-2},\phi)} \left[f(vg^{-2}b_0^{-2} - u(vg^{-2}b_0^{-2}, b_1)) + \overline{F_+}(u(vg^{-2}b_0^{-2}, gb_0h), gb_0h) \right] dQ_1(h) + ... \\
= \;& \int_0^{b(vg^{-2},1)} \left(\frac{1}{b_0} \left[\frac{1}{gh} + gh\left(\frac{1}{gh^2} - \widehat{u}(1, gh)\right) \right] + \overline{F_+}(\widehat{u}(1, gh), gh)b_0^{-1} \right) dQ_1(h) \\
& + \int_{b(vg^{-2},1)}^{b(vg^{-2},\phi)} \left[b_0^{-1} f(vg^{-2} - u(vg^{-2}, hg)) + \overline{F_+}(u(vg^{-2}, hg), hg)/b_0 \right] dQ_1(h) \\
& + ... \\
= \;& b_0^{-1} \int_0^{b(vg^{-2},1)} \left(\left[\frac{1}{gh} + gh\left(\frac{1}{gh^2} - \widehat{u}(1, gh)\right) \right] + \overline{F_+}(\widehat{u}(1, gh), gh) \right) dQ_1(h) \\
& + b_0^{-1} \int_{b(vg^{-2},1)}^{b(vg^{-2},\phi)} \left[(f(vg^{-2} - u(vg^{-2}, gh)) + \overline{F_+}(u(vg^{-2}, gh), gh) \right] dQ_1(h) + ... \\
= \;& \frac{1}{b_0}\gamma^{-1}F(vg^{-2}, g, \phi)
\end{aligned}$$

Taking averages we obtain the required result.

11 Appendix F: Book-value in the Feltham Ohlson model

It bears remarking here that the framework of the Feltham-Ohlson model takes as its primitive a notion of accounting valuation, namely the book-value (from which 'earnings' are defined once dividends are known). Properly speaking, implicit to the model is therefore a **valuation function** $b(.)$ defining the book value from the portfolio $H_t = (c, v_0, v_1, ..., v_t)$ of ex-dividend cash, c, and unused investment assets $v_0, ..., v_t$ where v_i was bought at times t_i and price b_i then the historic cost convention is that

$$B_t = c + v_0 b_0 + ... + v_t b_t,$$

that is the valuation takes the general form:

$$B_t = b(t, H_t)$$

However, the realisation of the abnormal earnings stream $\widetilde{N}_t = \{\tilde{v}_t\}$, as defined from $b(.,.)$, is then predicted by a model of its dynamics M, which typically depends upon the current book value as initial condition [32], and so implies first of all a stochastic process $\widetilde{M}_t = \{\tilde{v}_t^M\}$, i.e. stochastically generated prediction of the realised stream $\widetilde{N}_t$, and then the price of equity via the identity (3). Thus predicted price of equity is affected by the accounting convention (which is the historic cost convention in the Feltham Ohlson model). To see this more clearly suppose b' is an alternative accounting convention, yielding the alternative valuations

$$\begin{aligned} B'_t &= b'(t, H_t), \\ \tilde{v}'_t &\equiv v_t - rB'_{t-1}, \end{aligned}$$

then, provided B'_t also satisfies the standard technical assumption, we have, by the usual argument

$$B_t + \sum_{\tau=1}^{\infty} \gamma^\tau E_t(\tilde{v}_{t+\tau}) = W_t = B'_t + \sum_{\tau=1}^{\infty} \gamma^\tau E_t(\tilde{v}'_{t+\tau}).$$

So, abbreviating the summation of discounted expected values temporarily to V_t we have

$$B_t + V_t[\widetilde{N}_t] = B'_t + V_t[\widetilde{N'}_t]. \tag{37}$$

It could therefore be that the same model of the earnings stream dynamics M gives a value

$$B'_t + V_t[\widetilde{M'}_t]$$

which is a better predictor of W_t than $B_t + V_t[\widetilde{M}_t]$. Now observed actual discrepancies from the realization could either deny validity of the $AR(1)$ assumption, or require that any explanation absorb the discrepancy in a dividend policy consistent with the $AR(1)$ assumption via (2), i.e.

$$d_t^M = \tilde{v}_t^M - B_t + (1+r)B_{t-1}.$$

[32] By formulas like

$$P_t = y_t + \frac{\omega}{R-\omega}\tilde{v}_t + \frac{R}{(R-\omega)(R-\gamma)}x_t.$$

However, an alternative accounting convention could perhaps generate different model predictions closer to reality despite using the same underlying stochastic dynamics.

Evidently, the technical assumption proving (3), namely that $\gamma^\tau B_\tau \to 0$ as $\tau \to \infty$ (i.e. that book value does not grow faster than the bank yield $1+r$), implicitly favours the **historic cost convention** (as perpetually unused stock is in the limit discounted to zero). However, the technical assumption may be satisfied by an other convention governing unused production input assets provided, for instance, that these assets are utilized almost surely within a uniformly bounded horizon. In reality there is an expiry date for most inputs and this guarantees that it is optimal to utilize them ahead of the best-before date.

Fortunately no such technicalities arise in a finite horizon, and in that setting there is a identity corresponding to (3) that includes final book-value B_T (possibly as final dividend). There is thus an alternative convention directly justifiable by the definition of residual income itself. Inspection of an equivalent to the defining equation, namely

$$\widetilde{v}_t = B_t - (1+r)B_{t-1} + d_t \tag{38}$$

in which old book-value is interest-adjusted before being deducted from current book value suggest a common value rendering of the two book-values. We may therefore justifiably use as accounting valuation function $b'(.)$ by following the **common value accounting convention** so that

$$B'_t = c + v_0(1+r)^t b_0 + v_1(1+r)^{t-1} b_1 + ... + v_t b_t$$

and the valuation B'_t thus includes in c interest on cash in the bank from recorded earlier revenues and also attracts cost of capital charges on top of historic costs.

Thus cost of unused stock recorded in both B'_t and B'_{t-1} on this convention cancel each other out in the (38) calculation of residual income, allowing treatment of unused ‘stock in hand’ just like interest on any earlier cash deposits sitting in the bank. This has two important consequences:

(i) residual income attributable to immediate utilization of fresh stock is **increased** by comparison to the historic cost convention which would includes in addition the cost of unused capital;

(ii) residual income attributable to eventual utilization of long unused stock is **decreased** relative to the historic cost convention.

Both these factors properly reflect return from investment in rewarding the record of profitable activity from investment and down-playing unprofitable activity. See section below for a worked example. Note that any unused stock sold back will also increase the value of residual income as a cash addition.

We stress that both conventions must of necessity give rise to the same value of the firm by (37), and either earnings stream may be interpreted from the other, for instance

$$V_t[\widetilde{N}_t]_{\text{hist}} = (V_t[\widetilde{N}'_t] + B'_t)_{\text{common}} - (B_t)_{\text{hist}},$$

or as

$$(\widetilde{v}_t)_{\text{hist}} = (\widetilde{v}'_t)_{\text{common}} + [B_t - (1+r)B_{t-1}]_{\text{hist}} - [B'_t - (1+r)B'_{t-1}]_{\text{common}}.$$

However, as each gives a different interpretation to the term 'residual income', each offers a different route to predicting managerial activity and predicted residual earnings stream. In each dividends are still left outside the scope of equity value computation.

We should point out an additional advantage of the modified convention that well serves our purposes. If we employ a model of economic activity with constant expected return then common value convention automatically gives constant returns to unused stock.

11.0.1 Example: A stylised two period residual income model

Suppose we start with $x+u$ units of capital at $t=0$ purchased for p_0 a unit[33] and we plan to use x of the units in the first period and u of the units in the second period[34] with a square root returns function operating in both periods, that is:

$$\text{opening net assets } B_0 = p_0(u+x)$$

We assume a square root returns function.

Version 1: stylised model under historic cost convention We compute the two periods' respective earnings and residual incomes under the historic cost convention

$$\begin{array}{ll}
B_1 = 2\sqrt{x} + p_0 u & B_2 = (1+r)2\sqrt{x} + 2\sqrt{u} \\
B_1 - B_0 = v_1 & B_2 - B_1 = v_2 \\
v_1 = 2\sqrt{x} - p_0 x & v_2 = 2\sqrt{u} - p_0 u + 2\sqrt{x} r \\
\widetilde{v}_1 = v_1 - r p_0 (u+x) & \widetilde{v}_2 = v_2 - r(2\sqrt{x} + p_0 u) \\
= 2\sqrt{x} - (1+r)p_0 x - r p_0 u & = 2\sqrt{u} - (1+r)p_0 u
\end{array}$$

Note that the revenue $2\sqrt{x}$ included in B_1 arises at the end of the first period (i.e. time $t=1$). As a check, note the value of the firm at time $t=0$ is

$$\begin{aligned}
& B_0 + \frac{\widetilde{v}_1}{1+r} + \frac{\widetilde{v}_2}{(1+r)^2} \\
= \;& p_0(u+x) + \frac{2\sqrt{x} - (1+r)p_0 x - r p_0 u}{1+r} + \frac{2\sqrt{u} - (1+r)p_0 u}{(1+r)^2} \\
= \;& p_0 u + \frac{2\sqrt{x} - r p_0 u}{1+r} + \frac{2\sqrt{u} - (1+r)p_0 u}{(1+r)^2} \\
= \;& \frac{2\sqrt{x}}{1+r} + \frac{2\sqrt{u}}{(1+r)^2}.
\end{aligned}$$

[33] Assume this is financed by the owners initial equity investment.

[34] In order to make the simplist representation we shall assume that u is the dynamically optimal second period usage; that is even though the firm could buy or sell more units after observing the second period input price of capital it is not optimal to buy or sell capital. Our immediate object here is to map the the two models into a common notation rather than to concentrate on optimization. Once the mapping is established we will return to optimization issues.

Version 2: stylised model under 'common values' convention And now we compute using the common value accounting convention, as given below equ

ation (38):

$$\begin{array}{ll} B_1' = 2\sqrt{x} + p_0 u(1+r) & B_2' = (1+r)2\sqrt{x} + 2\sqrt{u} \\ B_1' - B_0' = v_1' & B_2' - B_1' = v_2' \\ v_1' = 2\sqrt{x} - p_0 x + p_0 ur & v_2' = 2\sqrt{u} - p_0 u(1+r) + 2\sqrt{x}r \\ \widetilde{v}_1' = v_1' - rp_0(u+x) & \widetilde{v}_2' = v_2' - r(2\sqrt{x} + p_0 u(1+r)) \\ = 2\sqrt{x} - (1+r)p_0 x & = 2\sqrt{u} - (1+r)^2 p_0 u \end{array}$$

Here

$$\begin{aligned} & p_0(u+x) + \frac{2\sqrt{x} - (1+r)p_0 x}{1+r} + \frac{2\sqrt{u} - (1+r)^2 p_0 u}{(1+r)^2} \\ = \quad & \frac{2\sqrt{x}}{1+r} + \frac{2\sqrt{u}}{(1+r)^2} \end{aligned}$$

Observe that B_1' includes the current income and the interest-adjusted historic valuation of unused stock left languishing; hence the residual income $\widetilde{v}_1'$ comprises the profit on current production using stock valued at the interest-adjusted historic valuation (as it was bought one period ago). Similarly, B_2' includes the current cash revenue and deposited cash revenues from the previous period (compounded up); consequent on the treatement in B_1' of unused stock, the residual income $\widetilde{v}_2'$ here equals the profit from final production using long unused stock valued at the interest-adjusted historic valuation (bought two periods ago). Recalling

$$(\widetilde{v}_t)_{\text{hist}} = (\widetilde{v}_t)_{\text{common}} + [B_t - (1+r)B_{t-1}]_{\text{hist}} - [B_t' - (1+r)B_{t-1}']_{\text{common}}.$$

we have

$$(B_1)_{\text{hist}} - (B_1')_{\text{common}} = (2\sqrt{x} + p_0 u) - (2\sqrt{x} + p_0 u(1+r)) = -p_0 ur$$
$$(B_2)_{\text{hist}} - (B_2')_{\text{common}} = 0$$

12 Appendix G: Alternative stylised models

We follow through the valuation formula in some simple instances. We begin with an important comment.

12.1 Comment on a Simple Example.

Consider the Cobb-Douglas model with production function $f(x) = 2\sqrt{x}$. (This example is characteristic of all the Cobb-Douglas models). Here $f^{\#}(x) = \sqrt{x}$ and $\{f'\}^{-1} \equiv G(b) = b^{-2}$. Observe first that the myopic one-period problem at time $t = t_m$ is to maximize

$$f(x) - p_m(1+r)x,$$

where p_n is the observed price, the factor $(1+r) = \gamma^{-1}$ represents an interest load on the cost payable at the end of the period. If the firm is in the black this is only the opportunity cost of capital. The above formalism assumes that there is no technical progress factor. Note that $f(x)$ represents payments received at the end of a production period. Thus the first order condition is

$$f'(x) = b_m$$

where

$$b_m = p_m(1+r).$$

Thus the optimal demand for input is $x^* = G(b_m)$ and so the (indirect) profit[35] is

$$f(x^*) - b_m x^* = f^{\#}(G(b_m)) = b_m^{-1}.$$

12.2 Two period model with a capital stock of u at the end of the first period

In the two period model we have:

$$\begin{aligned}
V_1(u, b_0, \phi_1) &= \int_0^{b_1(u)} (\frac{1}{b_1} + b_1 u) dQ(b_1) + 2\sqrt{u} \int_{b_1(u)}^{b_1(u)/\phi_1} dQ(b_1) + \int_{b_1(u)/\phi_1}^{\infty} (\frac{1}{\phi_1 b_1} + u\phi_1 b_1 dQ(b_1) \\
&= \int_0^{b_1(u)} \frac{1}{b_1} dQ(b_1) + \int_{b_1(u)}^{b_1(u)/\phi_1} \sqrt{u} dQ(b_1) + \int_{b_1(u)/\phi_1}^{\infty} \frac{1}{\phi_1 b_1} dQ(b_1) \\
&\quad + u\left[\int_0^{b_1(u)} b_1 dQ(b_1) + \frac{1}{\sqrt{u}} \int_{b_1(u)}^{b_1(u)/\phi_1} dQ(b_1) + \int_{b_1(u)/\phi_1}^{\infty} \phi_1 b_1 dQ(b_1)\right] \\
&= \int_0^{b_1(u)} f^*(G(b_1)) dQ(b_1) + \int_{b_1(u)}^{b_1(u)/\phi_1} f^*(u) dQ(b_1) + \int_{b_1(u)/\phi_1}^{\infty} f^*(G(\phi_1 b_1)) dQ(b_1) \\
&\quad + u\left[\int_0^{b_1(u)} b_1 dQ(b_1) + b_1(u) \int_{b_1(u)}^{b_1(u)/\phi_1} dQ(b_1) + \int_{b_1(u)/\phi_1}^{\infty} \phi_1 b_1 dQ(b_1)\right] \\
&= E[f^*(x_1^*(b_1))] + uq(b_1(u)).
\end{aligned}$$

[35] This is of course the Fenchel dual

$$\widetilde{f}(b) = \max_x [f(x) - bx].$$

Recall $b_1(u)$ is known at time t_0. Thus the future value derived from the project given a sunk resource u equals the Tobin valuation of the resource plus the expected indirect profit. In particular if $u = 0$ we have

$$V_1(0, b_0) = E[f^*(x_1^*(b_1))] = \int_0^\infty \frac{1}{b_1} dQ(b_1)$$

and if $u = \widehat{u}_1$ we have

$$\begin{aligned} V_1(\widehat{u}_1, b_0) &= E[f^*(x_1^*(b_1))] + \widehat{u}_1 q_0(b_1(\widehat{u}_1)) \\ &= E[f^*(x_1^*(b_1))] + \widehat{u}_1 b_0. \end{aligned}$$

If the project begins with zero stock and carries forward u and the charge on the capital u is b_0 then the clean surplus is

$$f^\#(G(b_0)) + \gamma V_1(u, b_0) - b_0 u = f^\#(x_0^*) + \gamma E[f^*(x_1^*(b_1))] + \widehat{u}_1[\gamma q(b_1(u)) - b_0],$$

and if the amount u is selected optimally at $\widehat{u}_1$ we have $\gamma q(b_1(\widehat{u}_1)) = b_0$ and so the clean surplus is

$$f^\#(x_0^*) + \gamma E[f^*(x_1^*(b_1))].$$

12.3 Two period model with 0 capital stock at the beginning of the first period

First consider time $t = t_0$ with zero stock. The optimal stock to purchase is

$$\widehat{v}_0 = \frac{1}{b_0^2} + \widehat{u}_1(b_0)$$

and this is bought for a price b_0 (valued at end of period time t_1, i.e. $b_0 = p_0(1 + r)$ where p_0 is the price announced/revealed at time t_0). From this investment the project generates immediate revenues of

$$\frac{2}{b_0}$$

and offers an additional expected PV (=present value)

$$\gamma V_1(\widehat{u}_1, b_0).$$

In the two period case we have

$$V_1(\widehat{u}_1, b_0) = E[f^*(x_1^*(b_1))] + \widehat{u}_1 b_0,$$

and in general

$$\begin{aligned} V_1(\widehat{u}_1, b_0) &= E_{b_1}[f^\#(x^*(\widehat{u}_1, b_1)) + \gamma \overline{V_2}^\#(u^*(\widehat{u}_1, b_1), b_1)] + \widehat{u}_1 b_0 \\ &= \widehat{u}_1 b_0 + E[\sum_{n=1}^N \gamma^{n-1} f^\#(x_n^*(b_n))]. \end{aligned}$$

Let $V(\widehat{v}_0)$ be the present value of the project (at time t_1) net of $\widehat{v}_0$ then

$$\begin{aligned} V(\widehat{v}_0) &= f(x_0) + \gamma V_1(\widehat{u}_1, b_0) \\ &= f(x_0) + \gamma q_0 \widehat{u}_1 + E[\sum_{n=1}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))] \end{aligned}$$

and inclusive of costs we have clean profit equal to

$$V_0(\widehat{v}_0) - b_0 \widehat{v}_0 = f^{\#}(x_0) + E[\sum_{n=1}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))].$$

Here is an alternative derivation. Begin with a valuation as follows:

credits less debits

where credits comprise: revenues + discounted valuation of future indirect profits +book-value of stock, whereas debits consist entirely of capital charge. The valuation term thus assumes the stock carried forward is a sunk cost. Now $\gamma q_0 = b_0$ (noting that $b_0 = \gamma p_0$, the price at time t_0 so that in fact we have $q_0 = p_0)$. Here we have that the PV of the project net of $\widehat{v}_0$ is

$$V(\widehat{v}_0) = f(x_0) + \gamma V_1.$$

So the clean surplus is

$$\begin{aligned} V(\widehat{v}_0) - b_0 \widehat{v}_0 &= f^{\#}(x_0) + [\gamma q_0 - b_0] \widehat{u}_1 + E[\sum_{n=1}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))] \\ &= f^{\#}(x_0) + E[\sum_{n=1}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))]. \end{aligned}$$

Thus the valuation on the rhs contains the PV of the charge for the sunk cost of $\widehat{u}_1$.

12.4 Two period model with initial stock 0 at time t_0, and understocked at time t_1

Given the initial stock of zero the project commences the period $t = t_1$ with stock $\widehat{u}_1$. Here we suppose that the optimal stock to hold is greater than $\widehat{u}_1$ and is

$$\widehat{v}_1 = \frac{1}{b_1^2} + \widehat{u}_2(b_1).$$

Additional stock required is thus

$$\widehat{v}_1 - \widehat{u}_1 = \frac{1}{b_1^2} + \widehat{u}_2(b_1) \frac{1}{b_1^2} - \widehat{u}_1(b_0).$$

The project generates new revenue

$$\frac{2}{b_1}$$

thus the present value of the project at time t_1 is as before:

credits less debits

where credits comprise: past and present revenues + discounted valuation of 'continued activity' (meaning future indirect profits with stock carried forward $\widehat{u}_2$ treated as a sunk cost) +book-value of stock, whereas debits consist entirely of capital charge. Here we have, since $\gamma q_1 = b_1$ that the PV is

$$\begin{aligned} & \frac{2}{b_0} + \frac{2\gamma}{b_1} + E[\sum_{n=2}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))] + \gamma^2 q_1 \widehat{u}_2(b_1) \\ & -b_0(\frac{1}{b_0^2} + \widehat{u}_1(b_0)) - \gamma b_1(\frac{1}{b_1^2} + \widehat{u}_2(b_1) - \widehat{u}_1(b_0)) \\ = \ & \frac{1}{b_0} + \frac{\gamma}{b_1} + \gamma^2 V_2(\widehat{u}_2) + \gamma b_1 \widehat{u}_2(b_1) - b_0 \widehat{u}_1(b_0) - \gamma b_1(\widehat{u}_2(b_1) - \widehat{u}_1(b_0)) \\ = \ & f^{\#}(x_0) + \gamma f^{\#}(x_1) + E[\sum_{n=2}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))] - (b_0 - \gamma b_1)\widehat{u}_1(b_0). \end{aligned}$$

Thus the value of the project comprises the realized indirect profits plus the future valuation of indirect profits less an adjusted charge on the initial forward outlay.

Alternatively derivation: we have, given a sunk cost $\widehat{u}_2$

$$\begin{aligned} V_2(\widehat{u}_2, b_1) &= E_{b_2}[f^{\#}(x^*(\widehat{u}_2, b_2)) + \gamma \overline{V_3}^{\#}(u^*(\widehat{u}_1, b_1), b_1)] + \widehat{u}_2 q_1 \\ &= \widehat{u}_2 q_1 + E[\sum_{n=2}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))]. \end{aligned}$$

so we have current plus expected revenues

$$V(\widehat{v}_0) = f(x_0) + \gamma f(x_1) + \gamma^2 V_2(\widehat{u}_2, b_0)$$

less costs

$$b_0 \widehat{v}_0 + \gamma b_1[\widehat{v}_1 - \widehat{u}_1]$$

giving clean profit of:

$$\begin{aligned} & f^{\#}(x_0) + \gamma f^{\#}(x_1) + \gamma^2[\widehat{u}_2 q_1 + E[\sum_{n=2}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))]] \\ & -b_0 \widehat{u}_1 - \gamma b_1[\widehat{u}_2 - \widehat{u}_1] \\ = \ & f^{\#}(x_0) + \gamma f^{\#}(x_1) + E[\sum_{n=2}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))] - \widehat{u}_1[b_0 - \gamma b_1]. \end{aligned}$$

12.5 Three period model initial stock 0 at time t_0, and understocked at time t_1 and again at time t_2

Given the stated assumptions at time $t = t_2$ the stock on opening is equal to $\widehat{u}_2$ and is less than optimal stock $\widehat{v}_2$. Buy extra $\widehat{v}_2 - \widehat{u}_2$ at price b_2. Here we have net of $\widehat{v}_2$ new revenues of

$$W = f(x_2) + \gamma V_3(\widehat{u}_3, b_2)$$

So

$$\begin{aligned} W - b_2\widehat{v}_2 &= f^{\#}(x_2) + \gamma q_2\widehat{u}_3 + E[\sum_{n=3}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))] - b_2\widehat{u}_3 \\ &= f^{\#}(x_2) + E[\sum_{n=3}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))] \end{aligned}$$

Hence all revenues amount to

$$f(x_0) + \gamma f(x_1) + \gamma^2 f(x_2) + \gamma^3 V_3(\widehat{u}_3, b_2)$$

with total costs being

$$b_0\widehat{v}_0 + \gamma b_1[\widehat{v}_1 - \widehat{u}_1] + \gamma^2 b_2[\widehat{v}_2 - \widehat{u}_2]$$

leading to a profit of

$$\begin{aligned} & f^{\#}(x_0) + \gamma f^{\#}(x_1) + \gamma^2 f^{\#}(x_2) + E[\sum_{n=3}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))] + \gamma^3 q_2\widehat{u}_3 \\ & -\{b_0\widehat{u}_1 + \gamma b_1[\widehat{u}_2 - \widehat{u}_1] + \gamma^2 b_2[\widehat{u}_3 - \widehat{u}_2]\} \\ = \; & f^{\#}(x_0) + \gamma f^{\#}(x_1) + \gamma^2 f^{\#}(x_2) + E[\sum_{n=3}^{N} \gamma^{n-1} f^{\#}(x_n^*(b_n))] + \\ & -\{[b_0 - \gamma b_1]\widehat{u}_1 + \gamma[b_1 - \gamma b_2]\widehat{u}_2\}. \end{aligned}$$