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Abstract. A minimizing Searcher S and a maximizing Hider H move at unit speed on a closed interval until the first (*capture*, or *payoff*) time $T = \min\{t : S(t) = H(t)\}$ that they meet. This zero-sum *Princess and Monster Game* or less colorfully *Search Game With Mobile Hider* was proposed by Rufus Isaacs for general networks Q. While the existence and finiteness of the value $V = V(Q)$ has been established for such games, only the circle network has been solved (value and optimal mixed strategies). It seems that the interval network $Q = [-1, 1]$ had not been studied because it was assumed to be trivial, with value $3/2$ and 'obvious' searcher mixed strategy going equiprobably from one end to the other. We establish that this game is in fact non-trivial by showing that $V < 3/2$. Using a combination of continuous and discrete mixed strategies for both players, we show that $15/11 \leq V \leq 13/9$. The full solution of this very simple game is still open, and appears difficult, though many properties of the optimal strategies are derived here.

1. Introduction

In the final chapter of his classic book Differential Games [18], Rufus Isaacs introduced search games with mobile hiders, which he also called Princess and Monster Games (see also [7], example 1.4). A Searcher (Monster) and a Hider (Princess) move about a space Q, which we take to be a network (and later specialize to an interval). The Searcher chooses as his pure strategy a path $S = S(t)$ of known speed, which we take to be 1. He says "We permit the princess full freedom of locomotion", which we take to be any continuous path $H = H(t)$. (We will establish for the interval that she need never go faster than the Searcher.) The payoff for this zero-sum game $\Gamma(Q)$ is the capture time

$$T = T(S, H) = \min\{t : S(t) = H(t)\}. \tag{1}$$

Taking the topology of uniform convergence on compact subsets, the payoff function T is upper semi-continuous and the searcher mixed strategy space is compact Hausdorff. Consequently the minimax theorem of Alpern and Gal [4] can be used, as shown in Appendix A of [6] or [14] to establish the existence of the value $V(Q)$, an optimal mixed searcher strategy, and an ε-optimal hider mixed strategy. Recall that a strategy is ε-optimal if the expected payoff is at least $V - \varepsilon$ against any strategy of the opponent. Upper bounds on $V = V(Q)$ in terms of the structure of Q (and hence the finiteness of V) are derived in [3]. For general networks Q, it is sometimes advantageous for the Searcher to wait for a while at a node, a so-called *ambush strategy*, and these games are known to be difficult - none have been solved. So the only networks that might appear possible to solve are those with no nodes (of degree greater than 2) - namely the circle and the interval. The game $\Gamma(Q)$ when Q is a circle was indeed a problem suggested by Isaacs [18, Example 12.4.2], and was solved a long time ago ([23],[1]). The solution, for both players, is the *cohatu* strategy: start randomly (uniform distribution); flip a coin; if head (tails) go to

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antipodal point half way around circle clockwise (anti-clockwise), at unit speed. No one seems to have considered this problem for the other network without nodes, the interval. It seems to have generally been believed that the game on the interval was trivial. The Searcher should simply start at a random end and go directly to the other end. (Against this, the hider waits at 0 until time $1-\varepsilon$, then goes equiprobably to either end.)
It should be noted that this 'simple' search strategy, of starting at a random end and going to the other, is indeed optimal in the related *search game with immobile hider*, also introduced by Isaacs [18, Section 12.3]. Indeed, for trees [11] and trees with Eulerian networks attached [2], an optimal search strategy is to traverse a Chinese Postman path (minimal covering path) equiprobably in either direction. Of course this is obvious for the interval, where the Hider can hide uniformly or equiprobably at the ends (or many other optimal strategies - the full class has not been determined). The value is 1 for this game on an interval of length 2. A related problem on the interval, also with two mobile agents (they both have unit speed), was analyzed by Howard [17] and by Chester and Tutuncu [10]. In this rendezvous version of the problem, both players wish to *minimize* the meeting time T.
The paper is organized as follows. In Section 2 we establish some elementary lemmas which restrict the strategies we need to consider in the rest of the paper. In Section 3 we show that the game $\Gamma(I)$ is not trivial by using some finitely supported mixed strategies to obtain estimates on it's value V. In Section 4 we obtain the bound $V \leq 13/9$ by using a continuous mixed searcher strategy. In Section 5 we obtain the bound $V \geq 15/11$ by using a continuous mixed hider strategy.

2. Properties of optimal strategies

In this section we present results which restrict the strategies (pure and mixed) that we will need to consider in the remainder of the paper. We begin by noting that the pure strategy space for the Hider is the space $\mathcal{H}$ consisting of all continuous paths $H: [0,\infty) \to I = [-1,1]$. For the Searcher the pure strategy space $\mathcal{S}$ consists of all paths in $\mathcal{H}$ with Lipschitz constant 1, that is

$$\mathcal{S} = \left\{S\colon [0,\infty) \to I\ ;\ \left|S(t) - S(t')\right| \leq \left|t-t'\right|,\ \text{for all } t,t' \geq 0\right\}. \tag{2}$$

If a Searcher chooses paths that do not cover the entire interval, then hiding at some end gives a payoff that is infinite. This is absurd, so we may assume that searcher paths are onto. We show that once the Searcher reaches an end, he should go directly to the other end; but if the Hider reaches an end, he should stay there. The Hider, while unrestricted in speed, need never go faster than speed 1 (the Searcher's maximum speed). We show that both players can optimally use mixed strategies which respect the symmetry of the interval, that is, the reflection $\phi(x) = -x$. and that in an optimal hider mixed strategy, the pure strategies do not intersect. Finally, we give some properties of optimal response searcher strategies.

We say that a pure strategy (path) S is *end-reflecting* if whenever $S(t_0) = \pm 1$, we have $|S(t) - S(t_0)| = t - t_0$, for $t_0 \leq t \leq t_0 + 1$. We say that a pure strategy H is *end-absorbing* if $H(t_0) = \pm 1$ implies $H(t) = H(t_0)$ for $t \geq t_0$. We say that a mixed strategy (for either player) is *symmetric* if it is invariant under the reflection $\phi(x) = -x$.

The first three lemmas concern pure strategies which we may ignore in our subsequent analysis because they are dominated.

Proposition 1. *Every pure searcher strategy $S \in \mathcal{S}$ is dominated by one which is end-reflecting.*

Proof. Suppose S is not end-reflecting, and reaches say $+1$ at first time t_0. Define S^* as S up to time t_0 and then equal to $1+t_0-t$. Consider any $H \in \mathcal{H}$. If $T(S,H) \leq t_0$, then $T(S^*,H) = T(S,H)$. If $T(S,H) = t_1 > t_0$, then $H(t_1) - S^*(t_1) \geq 0$, and since H is continuous and $H(t_0) - S^*(t_0) = H(t_0) - 1 \leq 0$ the Intermediate Value Theorem implies that $T(S^*,H) = t_2$ for some t_2 with $t_0 \leq t_2 \leq t_1$. Thus in all cases the end-reflecting strategy S^* satisfies $T(S^*,H) \leq T(S,H)$. □

Proposition 2. *Every pure hider strategy H is dominated by one which is end-absorbing.*

Proof. Assume that H is not end-absorbing and arrives at say $+1$ at first time t_0. Let H^* be the end-absorbing strategy that agrees with H up to t_0 and then stays at $+1$. Consider any $S \in \mathcal{S}$ and assume, as we may, that $T(S,H^*) = t_1 > t_0$. Then $S(t_1) = 1 \geq H(t_1)$ and $S(t_0) < 1 = H(t_0)$ so the Searcher meets the Hider H between t_0 and t_1 by the Intermediate Value Theorem. It follows that $T(S,H) \leq T(S,H^*)$ for any pure searcher strategy. □

We say that a continously differentiable function is *smooth*. By reasons that will become clear below, we may restrict the pure hider strategies to any subset that is dense in $\mathcal{S}$. In particular, we may consider smooth paths only, without changing the value of the game.

Proposition 3. *Every smooth hider strategy in $\mathcal{H}$ is dominated by one in $\mathcal{S}$, that is, one with speed bounded by* 1.

Proof. Essentially, the idea is that if H is any smooth Hider and if H^* is a Hider that follows H but has speed bounded by 1, then H^* cannot be caught from behind since the Searcher has the same speed limit. Let $t_0 = \inf\{t \in [0,\infty) \colon |H'(t)| > 1\}$ and assume, as we may, that t_0 is finite. Define $H^*(t) = H(t)$ for $t \leq t_0$. For $t > t_0$ the Hider H^* continues to move at speed 1. Since the interval is bounded, H^* meets H at some time $\tau > t_0$. Now let $t_1 = \inf\{t \in [\tau,\infty) \colon |H'(t)| > 1\}$ and repeat the construction inductively.
Suppose that a Searcher S finds H at a time $T = T(S,H)$ when the hiders H and H^* are in different locations. So suppose that $t_0 < T < \tau$. By symmetry, we may assume as well that $H'(t_0) = +1$. Then H^* has velocity $+1$ and $H^*(t) < H(t)$ for all $t \in (t_0,t_1)$. Since $S(T) = H(T) > H^*(T)$ and since the Searcher moves with bounded speed, $S(t) > H^*(t)$ for all $t \in (t_0,T)$. This implies that $T(S,H^*) > T(S,H)$. □

From now on, we shall only consider hider paths of speed ≤ 1. We say that a Hider or a Searcher **runs** at time t if $|H'(t)| = 1$ or $|S'(t)| = 1$, respectively. We now consider mixed strategies; i.e, probability measures on the Borel σ-algebra of $\mathcal{S}$

Proposition 4. *There is an optimal searcher mixed strategy, and (for any ε) an ε-optimal hider mixed strategy, which are invariant under the reflection $\phi(x) = -x$.*

Proof. This is just a special case of Theorem 3 of Alpern and Asic [3], where the existence of such strategies invariant under the isometry group (distance-preserving homeomorphisms) of a network Q is established. In the case of $Q = I$, this group consists just of the identity and ϕ. □

The capture time $T(S,H)$ is upper semi-continuous as a function on the pure strategies $H \in \mathcal{H}$ and $S \in \mathcal{S}$. This implies that for any ε there exists a ε-optimal *finite* mixed strategy and in principle it is possible to determine the value of the game by considering finite mixed strategies only.

Proposition 5. *Suppose the Hider is using a mixed strategy concentrated on a finite number of pure strategies $H_j \in \mathcal{S}$, $j = 1,\dots,J$. Then any optimal response $S(t)$ by the Searcher is of the following* *type, for some renumbering of the H_j: It has capture times $T(S,H_j)=t_j$, with $t_1\le t_2\le\cdots\le t_J$, where $t_1=0$ and*

$$S'(t)=sign\,(H_{j+1}(t_j)-S(t_j))\,,\ t_j \tag{3}$$

That is, S moves at maximum speed 1 towards H_{j+1} as soon as he has met H_j.

Proof. Let S^* be any optimal response to the hider mixed strategy which fails (3) for some j. Number the H_j so that $T(S^*,H_j)=t_j^*$ is nondecreasing in j. Suppose that $j=k$ is the smallest j such that (3) fails. Suppose without loss of generality that $H_{k+1}(t_k^*)>S^*(t_k^*)$. Define a new searcher strategy $S(t)$ with capture times $t_j\equiv T(S,H_j)$ by

$$S(t)=\begin{cases} S^*(t)\,, & \text{if } t\le t_k^*,\\ S^*(t_k^*)+(t-t_k^*)\,, & \text{if } t_k^*\le t\le t_{k+1},\\ H_{k+1}(t) & \text{if } t_{k+1}\le t\le t_{k+1}^*\\ S^*(t)\,, & t_{k+1}^*\le t. \end{cases} \tag{4}$$

Clearly $t_j=t_j^*$ except for $j=k+1$, when $t_{k+1}^*>t_{k+1}$. Hence S^* could not have been an optimal response. (Note that the third definition $S(t)=H_{k+1}(t)$ is only possible because of our hypothesis $H_{k+1}\in S$, that is, it moves with speed bounded by 1.) □

This result is a simplified version of a similar observation for rendezvous on the line due to Alpern and Gal [5] (repeated as Theorem 16.10 of [6]). Basically, it says that an optimal response to a finitely supported hider mixed strategy is to run all the time and turn only when meeting one of the pure hider strategies. In particular, we can restrict our attention to pure strategies in which the Searcher runs all the time. In a space-time diagram $[-1,1]\times[0,\infty)$, such search paths are depicted as broken lines of slope ±1 with finitely many turning points, cf Figure 1 below.

Definition 6. *A pair of pure hider strategies H_1,H_2 is called* **non-crossing** *if (possibly after reordering) we have $H_1(t)\le H_2(t)$ for all $t\ge0$. If the inequality holds strictly, we say that they are* **non-intersecting**.

For pure hider strategies H_1 and H_2, define new pure strategies $H_1\wedge H_2(t)=\min\{H_1(t),H_2(t)\}$ and $H_1\vee H_2(t)=\max\{H_1(t),H_2(t)\}$. Obviously, $H_1\wedge H_2$ and $H_1\vee H_2$ are non-crossing.

Proposition 7. *The hider strategy that mixes two pure strategies H_1,H_2 with equal probability is dominated by the non-crossing hider strategy that mixes $H_1\wedge H_2, H_1\vee H_2$ with equal probability. Consequently, any finite mixed hider strategy may be assumed to consist of non-crossing pure strategies.*

Proof. Note that at for all t the sets $\{H_1(t),H_2(t)\}$ and $\{H_1\wedge H_2(t),H_1\vee H_2(t)\}$ are the same. So if S catches the first of the two original hiders H_1,H_2, then at the same time he catches the first of the non-crossing hiders $H_1\wedge H_2,H_1\vee H_2$. Denote this time by t_1. By renumbering indices or reflecting the interval we may assume that $S(t_1)=H_1(t_1)$ and that $H_1(t_1)\le H_2(t_1)$. Now under these assumptions, H_2 and $H_1\vee H_2$ are in the same location at time t_1 and the Searcher is to their left. Since $H_1\vee H_2(t)\ge H_2(t)$ for all t, the Searcher cannot catch $H_1\vee H_2$ before he catches H_2. □

It follows from Proposition 4 that there exist non-crossing ε-optimal hider strategies that are symmetric. Any finite collection of non-crossing paths can be approximated arbitrarily closely by a collection of non-intersecting paths. So, there exist ε-optimal mixed hider strategies that are finite, symmetric and non-intersecting.

Proposition 8. *Any pure strategy H in a non-intersecting symmetric hider strategy is contained in half of the interval, that is, $H(t) \in [-1, 0]$ or $H(t) \in [0, 1]$ for all t.*

Proof. If the pure strategy H is used in a symmetric mixed strategy, then so is $-H$. If $H(t) = 0$ then H and $-H$ intersects $-H(t)$, but the mixed strategy is non-intersecting. So either $H(t) \neq 0$ for all t or H is immobile and remains in 0. In both cases, H is contained in half of the interval. □

3. The Interval Game is Not Trivial

The interval game has some fairly obvious strategies for each player that appear like they might be optimal. If any of these were indeed optimal, we would consider the game to be trivial. The purpose of this section is to show that none of these strategies are in fact optimal. For the Searcher, the obvious strategy is to start at a random end and run to the other end; this gives an estimate $V \leq 3/2$ and the bound can be obtained if the Hider waits at the center until time $1 - \varepsilon$ and then runs to a random end. For the Hider, the two stationary strategies may be considered that are optimal in the immobile version of the game: one of these is to wait at a random end; the other is to wait randomly along the interval. Both guarantee $V \geq 1$, where 1 is the value of the immobile hider game. We show that the game is not trivial by exhibiting fairly simple strategies establishing that

$$1 < \frac{97}{75} < V < \frac{47}{32} < \frac{3}{2}.$$

Suppose that the Hider starts out with the strategy of hiding at an end point $E(t) = 1$ and symmetrically $-E(t) = -1$. According to Lemma 5 the optimal response of the Searcher is to start at an end point and run to the other end $A(t) = 1 - t$, or symmetrically $-A(t) = -1 + t$. The strategies A and $-A$ are what Howard [17] called the *sweepers* in his rendezvous version. The expected meeting time for this optimal response is 1 and this is a lower bound on V. Similarly, suppose that the Searcher adopts the sweeper strategies A and $-A$. Against any hider path H either $V(A, H) \leq 1$ or $V(-A, H) \leq 1$. So if the searcher adopts the sweeping strategy, then the payoff is $\leq 3/2$ against any mixed hider strategy and this puts an upper bound on V. The game would be trivial if $V = 1$ or $V = 3/2$, but it is not. In this section we show that $1 < V < 3/2$. It is ε-optimal against the sweeping strategy to loiter around 0 until time $1 - \varepsilon$ and then run to one of the end points. The Searcher can ambush such loiterers by adding a search path that patrols the centre. More specifically, strategy B starts at 0; runs to the left; follows sweeper A from the time $(1/2)$ when he meets him, until reaching $+1$; then (by Lemma 1) goes to -1. Strategies $\pm A$ are each used with probability 7/16, while $\pm B$ are each used with probability 1/16. The searcher strategies $\pm A, \pm B$ are drawn in a space-time diagram in Figure 1.

We now show that this mixed strategy ensures a meeting time less than 3/2 (though admittedly not much less).

Theorem 9. $V \leq \dfrac{47}{32} = 1.468\,8$

Proof. Consider the mixed strategy s in which the Searcher uses $\pm A$ each with probability 7/16 and $\pm B$ each with probability 1/16. Let H be *any* Hiding strategy. Let $P(t)$ denote the probability that $T \leq t$. There are two cases: (1) $|H(1/2)| \leq 1/2$, and (2) $|H(1/2)| > 1/2$.

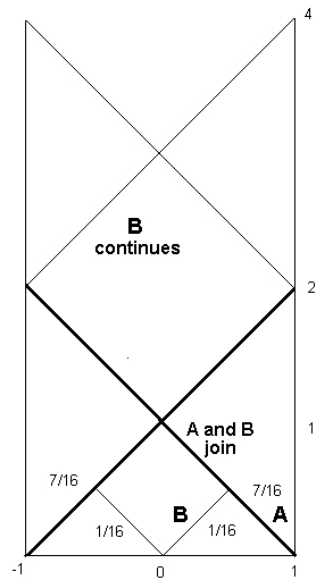

FIGURE 1. The searcher strategy for $V < 47/32$ in a space-time diagram

(i) In this case $P(1/2) \geq 1/16$ (because B or $-B$ has been met) and $P(1) \geq 8/16$ (because also A or $-A$ has been met). Furthermore $P(2) = 1$. Thus

$$T \leq \frac{1}{16} \cdot \frac{1}{2} + \frac{7}{16} \cdot 1 + \frac{1}{2} \cdot 2 = \frac{47}{32}$$

(ii) By symmetry we may assume $H(1/2) > 1/2$. Then $P(1/2) \geq 7/16$ (-A has been met), $P(2) \geq 15/16$ (A,-A,B have been met), and $P(4) = 1$ (all met). Hence

$$T \leq \frac{7}{16} \cdot \frac{1}{2} + \frac{8}{16} \cdot 2 + \frac{1}{16} \cdot 4 = \frac{47}{32}$$

□

We now consider the lower bound on V. In the search game with an immobile hider, the Hider has two particular mixed strategies that guarantee him an expected capture time of half the length of the interval: (1) Hide equiprobably at the ends (end-hiding is optimal on trees for such games [11], and symmetry implies the equiprobability), or (2) hide uniformly (this is optimal for all networks with Eulerian paths [2]). Mobility usually helps the Hider, for example when Q is the circle of circumference c an immobile hider can be found (by a random tour) in mean time $c/2$, while an mobile hider can be found with best play on both ends in time $3c/4$ [1]. The intuitive explanation is usually that 'when the Hider is immobile, the searcher does not have to search again any part of Q already searched (and hence can employ a minimal, Chinese Postman, search path) - but a *mobile* Hider might not be met by such a path. However this explanation does not apply to the interval (though to all other networks), since a Chinese Postman path on an interval will indeed find a mobile hider - it has search number 1 in the sense of Parsons [20].

Mobility helps the hider, since $V > 1$ if the Hider is mobile. To prove that this is true, we select hider strategies by considering the searcher strategies A and B. The optimal hider strategy against A, B is to loiter around $\pm\frac{1}{2}$ and just before time $\frac{1}{2}$ run either to the middle and back, or to the end. These two possible paths G, H and their symmetric counterparts are depicted in thick lines in the left-hand diagram in Figure 2 (the paths do not cross the centre). We combine these with the two other hider strategies that we considered above: E, hiding at an end, or F, loitering in the centre $F(t) = \max\{0, t+\varepsilon-1\}$. Then we get a mixed strategy in which the Hider uses $\{E, F, G, H\}$ and their symmetric counterparts. According to Lemma 5 the Searcher adopts strategies that start at $0, \pm\frac{1}{2} \pm 1$ and then run between hider paths. If the Searcher starts out from an end, then by Proposition 1 it is optimal to adopt strategy A. If he starts out in 0 then the Searcher runs to $\pm\frac{1}{2}$ at which point he may turn, 'strategy B', or continue to an end and run back, 'strategy M' in Figure 2. Starting from $\pm\frac{1}{2}$ the Searcher either runs to the centre or to an end, 'strategies C and D'. So the optimal optimal response to a mixed hider strategy on G and H is a mixed searcher strategy on $\{A, B, C, D, M\}$ and their symmetric counterparts. Ignoring ε these strategies give the

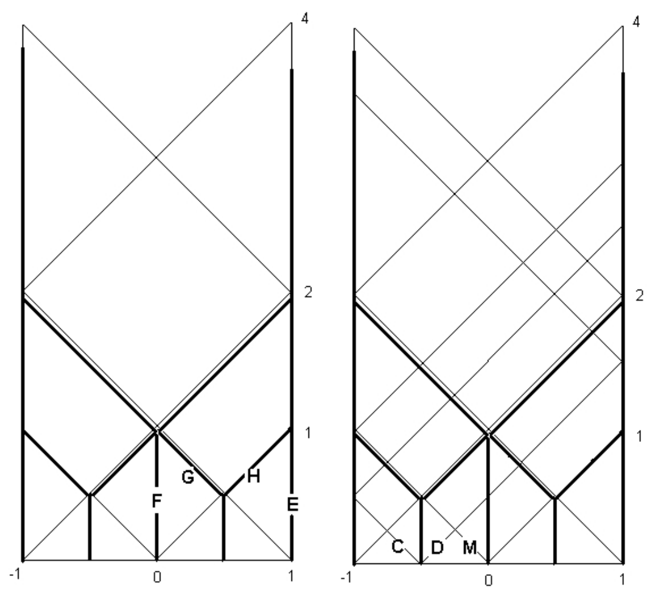

FIGURE 2. Left: the optimal hider strategy against A, B in a space-time diagram (recall that hider paths do not cross the middle); Right: the Searcher's response to $\{E, F, G, H\}$, up to symmetry the only relevant paths are A, B, C, D and M.

zero-sum game matrix:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ M \end{array} \begin{array}{c} \begin{array}{cccc} E & F & G & H \end{array} \\ \begin{bmatrix} 1 & \frac{3}{2} & \frac{3}{2} & \frac{5}{4} \\ 3 & 0 & \frac{5}{4} & 3 \\ \frac{3}{2} & \frac{15}{8} & \frac{5}{4} & \frac{5}{4} \\ \frac{5}{2} & \frac{1}{2} & \frac{3}{8} & \frac{3}{4} \\ 2 & 0 & \frac{7}{4} & 2 \end{bmatrix} \end{array} \tag{5}$$

It turns out that the Hider only uses strategies E, F, G and that the Searcher only uses A, D, M and of these he hardly ever uses M. The value of this matrix game $\frac{97}{75}$ puts a lower bound on V.

Theorem 10. $V \geq \frac{97}{75} = 1.29\,3$

It is possible to find finitely many hider strategies which optimize his response against A, B, C, D, M and this would give an upper bound on V. So, in principle, it is possible to construct in an iterative manner finite mixed strategies that put ever more accurate bounds on V. However, the number of pure strategies increases exponentially and the approximation of the true value of V appears to be very slow.

4. A searcher Strategy with a Continuous Initial Distribution

To improve on the upper bound $V < 47/32$ of Theorem 9, we extend the mixed searcher strategy which uses $\{A, B\}$. We replace the pure strategy B by a continuous mixed strategy s_Φ. In this strategy s_Φ the Searcher picks a point x according to a continuous distribution function on the interval $\Phi(x)$ and runs to the right until he meets the sweeper A, then joins the sweeper until he reaches -1 and runs back to the other end. The symmetric strategy $-s_\Phi$ starts according to $\Phi(-x)$ and the Searcher runs to the left until he meets the sweeper $-A$, etc.

Lemma 11. *Suppose that the Searcher uses the mixed strategy s_Φ. Let H be a pure hider strategy and let $y = y(H)$ be the first time that the Hider meets a sweeper. Then the Searcher finds the Hider before time y if and only if he starts in $(H(y) - y, H(0)]$ and runs to the right, or if he starts in $[H(0), H(y) + y)$ and runs to the left.*

Proof. By Proposition 8 we may assume that $H \geq 0$, so H meets the right sweeper A first and $H(y) = 1 - y$. We consider only the case that the searcher S initially runs to the right until he meets A. If S runs to the left the argument is the same. Suppose that s starts in $(1 - 2y, H(0)]$, so S meets A at time $t = \frac{1 - S(0)}{2} < y$. Then $S(0) < H(0)$ and $S(t) > H(t)$ since $H(t) < A(t)$ for $t < y$. Therefore S finds H in between time 0 and time t. Now suppose that S does not start in the interval $(1 - 2y, H(0)]$, so either $S(0) > H(0)$ or $S(0) < 1 - 2y$. In the first case, S runs towards A and finds the Hider at time y. In the second case, the Searcher cannot meet A at time y, while the Hider can. So the paths of S and H cannot cross before time y. □

Lemma 12. *Let $f = \Phi'$ be the probability density. Searchers that start in $(H(y) - y, H(0)]$ and run to the right catch the Hider with expected time*

$$\int_0^y tf\,(H(t) - t)\,(1 - H'(t))dt. \tag{6}$$

Searchers that start in $[H(0), H(y)+y)$ *and run to the left catch the hider with expected time*

$$\int_0^y tf\left(-H(t)-t\right)(1+H'(t))dt. \tag{7}$$

Proof. Consider a small time interval $[t, t+\Delta t]$ when the Hider moves from $H(t)$ to $H(t)+\Delta H$. Searchers that meet the Hider in that time interval and that have started out from the right, have started in $[H(t)-t+\Delta H-\Delta t, H(t)-t]$. Here we use that $|\Delta H| \leq \Delta t$ since we may assume that a Hider moves at no greater speed that 1. The probability of a Hider starting out in that interval is $f(H(t)-t)(\Delta H-\Delta t)$ up to first order. The time of capture is t up to first order. By taking the limit of $\Delta t \to 0$ we obtain the integral in (7). By symmetry we obtain (6). □

If a Hider has chosen $x = H(0)$ and y, then he maximizes the expected time of capture, which comes down to maximizing

$$\int_0^y t\left[f\left(H(t)-t\right)(1-H'(t)) + f\left(-H(t)-t\right)(1+H'(t))\right]dt. \tag{8}$$

This integral can be simplified by partial integration, which gives the sum of a constant $-y\Phi(1-2y)$ and the integral in equation (9).

Lemma 13. *Let* y *be the first time that the Hider meets a sweeper. The optimal Hider path from* $H(0)$ *to* $H(y)$ *maximizes*

$$\int_0^y \Phi\left(-t+H(t)\right) + \Phi\left(-t-H(t)\right)dt. \tag{9}$$

This is a variational problem. It's Euler-Lagrange equation is $f(-t+H(t)) = f(-t-H(t))$, where as before f is the probability density. A stationary value is $H(t)=0$ and this makes sense, since s_Φ is designed against loitering hiders and $H(t)=0$ is a minimum. If Φ is the uniform distribution, then the Euler-Lagrange equation is satisfied by any path and the integral does not depend on H: it is equal to $y(1-y/2)$.

The integral in (9) represents the payoff against the searchers that start in $(H(0)-y, H(0)+y)$ and run towards $H(0)$. Once the Hider has met a sweeper, he should run to the end since if the Searcher uses s_Φ he joins the sweeper. So we can determine the payoff $V(s_\Phi, H)$. Denote $x = H(0)$, then this payoff is:

$$1-\Phi(-x)+2\Phi(1-2y)+\frac{y}{2}(1-\Phi(x))-\frac{y}{2}\Phi(1-2y)+\frac{1}{2}\int_0^y \Phi(-t+H(t))+\Phi(-t-H(t))dt \tag{10}$$

This equation is derived as follows. A searcher starts out left from x and runs to the left with probability $(1-\Phi(-x))/2$. Such a searcher catches H at time 2. This gives the first term. A searcher starts out left from $1-2y$ and runs to the right with probability $\Phi(1-2y)/2$. Such a searcher catches H at time 4 and this gives the second term. A searcher starts out right from x and runs right with probability $(1-\Phi(x))/2$. Such a searcher catches H at time T. This gives the third term. The fourth term $-y\Phi(1-2y)$ turns up in the partial integration and the final term is the variational integral.

We consider a mixed strategy σ for the Searcher, as follows: use the sweeper strategy $\pm A$ with probability p and use the continuous mixed strategy s_Φ with probability $1-p$. If Φ is the uniform distribution then (10) is equal to

$$\frac{10+2x-(7+x)y+y^2}{4}, \tag{11}$$

which is maximal at $x=1$ for any $0 \le y \le 1$ and decreasing in y. If the Searcher takes $p = 7/9$ then $V(\sigma, H) = 13/9 - y/18 + y^2/18$ which is maximal at $y=0$ and $y=1$. So against σ, the Hider optimizes in either of two ways: stick to an end, or run from an end to the middle and back. The payoff is 13/9, which puts an upper bound on the value of the game that is sharper than the bound 47/32 that we found in the previous section. We summarize this in a theorem.

Theorem 14. *If the Searcher uses the mixed strategy σ then the optimal response of the Hider gives a matrix game with value* 13/9. *In particular $V \le 13/9$.*

Now the obvious way to try and improve on this bound is by varying the distribution Φ. Our computer experiments indicated that the bound of 13/9 can only be improved marginally in this way. To prove that there is only room for marginal improvement, we consider a specific non-crossing hider strategy: H_x starts in $H(0) = x \ge 0$ and runs towards the sweeper A; turning just ε in front of the sweeper; then H_x runs back towards the middle but turns once again at time y, before reaching the middle, to run to the end. So, ignoring ε we can describe this path by $H_x(t) = x+t$ if $t \le (1-x)/2$ and $H_x(t) = 1-t$ if $(1-x)/2 \le t \le y$ for $x \ge 0$. The variational integral (9) over this path is:

$$\frac{1}{2}\left(\Phi(x)(1-x) + \int_0^{-x} \Phi(t)dt + \int_{1-2y}^{x} \Phi(t)dt\right) \tag{12}$$

Suppose that $y=1$. Then the payoff $V(H_x, A)$ against the sweeper strategy is 3/2. Against the continuous strategy it is

$$V(H_x, s_\Phi) = \frac{3}{2} - \Phi(-x) - \Phi(x)\frac{(1+x)}{4} + \frac{1}{4}\int_{-1}^{-x}\Phi(t)dt + \frac{1}{4}\int_{-1}^{x}\Phi(t)dt \tag{13}$$

Suppose that the Searcher uses a mixed strategy σ_Φ on $\{A, s_\Phi\}$. Suppose that the Hider uses a mixed strategy γ on $\{E, H_1, H_{\frac{1}{2}}, H_0\}$, where E is the end point strategy. The integrals in (13) can be bounded from below by finite sums such as $\int_{-1}^{0}\Phi(t)dt \ge \frac{1}{2}\Phi(-\frac{1}{2}) + \frac{1}{2}\Phi(0)$. Bounding the integrals in this way we get a 4×2 zero-sum matrix game, the value of which is a lower bound on $V(\gamma, \sigma_\Phi)$:

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ 3 & 1+\frac{1}{8}\Phi(-\frac{1}{2})+\frac{1}{8}\Phi(0)+\frac{1}{8}\Phi(\frac{1}{2}) & \frac{3}{2}+\frac{1}{4}\Phi(-\frac{1}{2})-\frac{5}{4}\Phi(0) & \frac{3}{2}-\frac{7}{8}\Phi(-\frac{1}{2})+\frac{1}{8}\Phi(0)-\frac{3}{8}\Phi(\frac{1}{2}) \end{bmatrix} \tag{14}$$

To minimize the value of this matrix game, the Searcher should choose Φ such that the maximum of $\left\{1+\frac{1}{8}\Phi(-\frac{1}{2})+\frac{1}{8}\Phi(0)+\frac{1}{8}\Phi(\frac{1}{2}),\ \frac{3}{2}+\frac{1}{4}\Phi(-\frac{1}{2})-\frac{5}{4}\Phi(0),\ \frac{3}{2}-\frac{7}{8}\Phi(-\frac{1}{2})+\frac{1}{8}\Phi(0)-\frac{3}{8}\Phi(\frac{1}{2})\right\}$ is minimal. By linear programming we find that the Searcher should choose $\Phi(-\frac{1}{2}) = \frac{8}{25}$ and $\Phi(0) = \Phi(\frac{1}{2}) = \frac{9}{25}$. The value of the matrix game is $\frac{337}{237} = 1.4219\ldots$, which is only marginally smaller than 13/9.

Theorem 15. *If the Searcher uses a mixed strategy* σ_Φ *on* $\{A, s_\Phi\}$ *for any distribution* Φ *then the Hider can respond by a strategy* γ_Φ *such that* $V(\sigma_\Phi, \gamma) > 1.421$.

5. A hider Strategy with a Continuous Initial Distribution

To improve on the lower bound $V > 97/75$ of Theorem 10, we extend the mixed hider strategy which uses $\{E, F, G\}$. We replace the loitering strategies F and G by a continuous mixed strategy h_Φ. In this strategy the Hider picks a point $x \in [\varepsilon, 1-\varepsilon]$ according to a distribution function $\Phi(x)$ and he waits at x until the sweeper A is ε close. Then the Hider runs to ε, turns, and runs back to 1. In the symmetric strategy $-h_\Phi$ the Hider picks a point in $[-1+\varepsilon, -\varepsilon]$.

It follows from Proposition 5 that the Searcher S either starts at an end, in which case the S is a sweeper, or he starts in $[-1+\varepsilon, -\varepsilon] \cup [\varepsilon, 1-\varepsilon]$ Let y be the first time that S gets ε close to a sweeper. By symmetry we may assume that this sweeper is A. Since A runs all the time $S(t) < A(t)$ for $t < y$. We conclude that S approaches A from the left, so up to time y he catches hiders that have remained immobile. Clearly, the Searcher should maximize the interval $[S(0), S(y)]$ to catch as many immobile hiders as possible. So S starts in $1-2T-\varepsilon$ and runs to $1-T-\varepsilon$. At time y the Searcher catches all the loitering hiders that have started out from $x > S(0)$ and are running ε in front of A. The Searcher now effectively has two possibilities, and we leave it to the reader to check this: either he turns and runs to -1 and back to $+1$, let's call this S_1, or he continues and runs to $+1$ and then back to -1, let's call this S_2. Against the end point strategy the payoffs are $V(S_1, E) = 3$ and $V(S_2, E) = 1 + 2T$. Ignoring ε the payoffs of these two strategies against h_Φ are

$$\begin{aligned} V(S_1, h_\Phi) &= (1-\Phi(2y-1)) + \tfrac{1}{2}\int_0^{2y-1} f(t)dt + \frac{y(1-\Phi(1-y)) + \Phi(1-2y) + \int_{1-2y}^{1-y} f(t)dt}{2} \\ V(S_2, h_\Phi) &= (1+y)(1-\Phi(2y-1)) + \tfrac{1}{2}\int_0^{2y-1} f(t)dt + \frac{y(1-\Phi(1-y)) + (1+y)\Phi(1-2y) + \int_{1-2y}^{1-y} f(t)dt}{2} \end{aligned} \tag{15}$$

where, as before, f denotes the density of Φ. To see why this is true, note that the first two terms in $V(S_1, h_\Phi)$ concern loitering hiders that start out from $x < 0$: the term first represents the hiders that start from $x < 0$ and that are found at the end point -1, the second term represents the hiders that start from $1-2y < x < 0$, which are caught before time y. The third term represents hiders that start from $x > 0$. In the same way we obtain $V(S_1, h_\Phi)$. Both payoffs are functions of y.

We simplify the analysis and consider only the case that Φ is the uniform distribution. Let γ be the mixed strategy in which the Hider uses $\{E, h_\Phi\}$, with Φ equal to the uniform distribution.
Since $V(S_1, E)$ does not depend on T it is optimal for the Searcher to choose y such that $V(S_1, h_\Phi)$ is minimal.

Theorem 16. *If the Hider uses the mixed strategy* γ *then the optimal response of the Searcher gives a matrix game with value* $15/11$. *In particular* $V \geq 15/11$.

Proof. Since $V(S_1, E)$ does not depend on y, the Searcher should choose y such that $V(S_1, h_\Phi)$ is minimal. As is shown in Figure 3, the minimum is at $y = 1$. So the Searcher S_1 runs ε ahead of the sweeper and the payoff is $\frac{3}{4}$. If the Searcher only uses a mixed strategy on $\{A, S_1\}$, then we get a 2×2 matrix game $\begin{bmatrix} 1 & 3/2 \\ 3 & 3/4 \end{bmatrix}$ which has value $15/11$. In this game it is optimal for the Hider to choose the end point strategy E with probability $\frac{3}{11}$ and the loitering strategy h_Φ with probability $\frac{8}{11}$. It turns out that the Searcher cannot decrease the value of the game by including

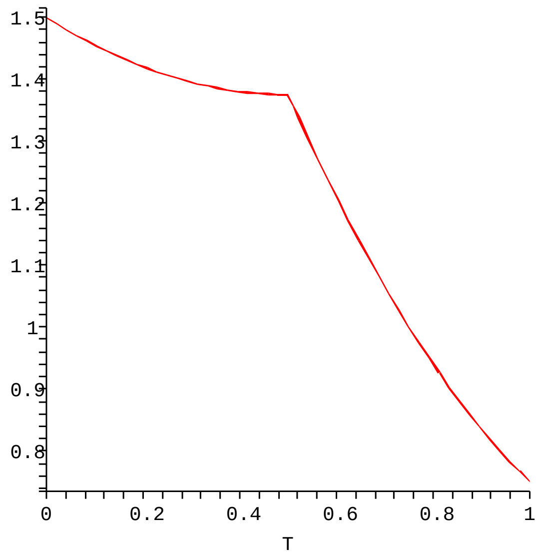

FIGURE 3. $V(S_1, h_\Phi)$ as a function of y

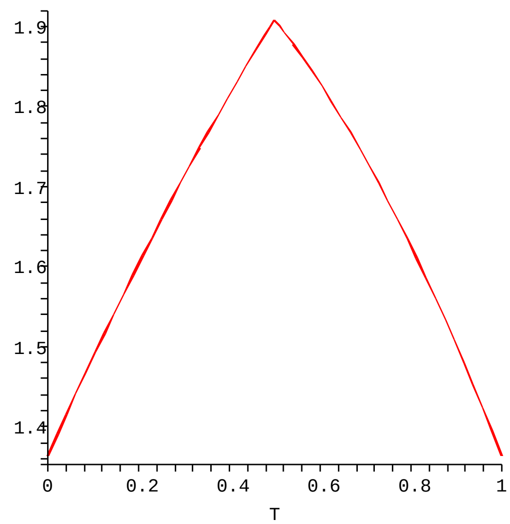

FIGURE 4. $\frac{3}{11}V(S_2, h_\Phi) + \frac{8}{11}V(S_2, h_\Phi)$ as a function of y

the strategy S_2: $\frac{3}{11}V(S_2, E) + \frac{8}{11}V(S_2, h_\Phi) \geq \frac{15}{11}$ as illustrated in Figure 4. So if the Hider uses the mixed strategy γ in which he chooses E with probability $\frac{3}{11}$ then the Searcher cannot do better than expected capture time 15/11. □

6. Conclusions

This paper introduces the apparently easy problem of how best to search for a mobile hider who is restricted to a known interval. The existence of a Value for this game, and of ε-optimal strategies for the Searcher and the Hider, follows from a minimax theorem by Alpern and Gal. However the determination of the value V, much less optimal strategies, seems difficult. The problem has resisted our attempts to solve it, but we have made significant progress in that direction. We have established many properties of optimal searcher and hider paths, that is, those that can be used in optimal mixed strategies. We have established bounds $15/11 \leq V \leq 13/9$ on the value of the game by developing a variational theory that can be used to evaluate certain mixed strategies which start according to a continuous distribution on the interval. We present this problem, the 'Princess and Monster Game on an Interval' as a challenge to the zero-sum game community. We conjecture that it's value is 1.4.

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